An Eberhard-like theorem for pentagons and heptagons

Robert Šámal

joint work with M. DeVos, A. Georgakopoulos, B. Mohar

Charles University, Prague
Simon Fraser University, Burnaby

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Euler formula

- Let G be a cubic plane graph,
- p_k ... number of its k-gonal faces.
- Euler's formula implies:

$$\sum_{k\geq 3}(6-k)p_k = 12.$$

Question

For which sequences $(p_k)_{k\geq 3}$ satisfying the above exists a cubic plane graph G whose face lengths are given by the sequence (p_k)?

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Not easy . . .

Theorem (B. Grünbaum, Convex polytopes, 1967)

- $p_4 = 6$, $p_6 \in \mathbb{N}$, *all other* $p_i = 0$ *. . . realizable iff* $p_6 \neq 1$*.*
- $p_5 = 12$, $p_6 \in \mathbb{N}$, *all other* $p_i = 0$ *. . . realizable iff* $p_6 \neq 1$*.*
- $p_3 = 4$, $p_6 \in \mathbb{N}$, *all other* $p_i = 0$ *. . . realizable iff* p_6 *is even.*

Adding hexagons

Theorem (Eberhard, 1891)

For every finite sequence $(p_k)_{k \neq 6}$ of non-negative integers satisfying Euler's condition, there are infinitely many values p_6 such that there exists a simple convex polyhedron having precisely p_k faces of length k for every $k \geq 3$.

(Simpler and complete proof in [B. Grünbaum, Convex polytopes, 1967] – using graphs instead of polyhedra.)

Main theorem

- want to understand better, what sequences are realizable
- modification: instead of adding hexagons we are adding C_5's and C_7's.
- if we add the same number of C_5's and C_7's, the equation $\sum_{k\geq 3}(6-k)p_k = 12$ remains valid.

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Main theorem – plane

Theorem (DeVos, Georgakopoulos, Mohar, Š., 2009)

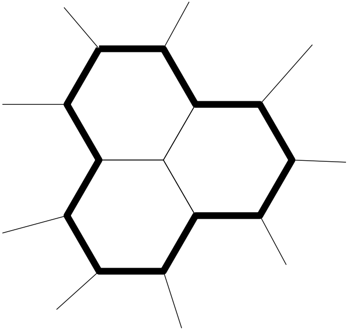
Let (p_k) be a finite sequence of non-negative integers satisfying the Euler condition. Then there exist infinitely many integers n such that after increasing p_5 and p_7 by n we obtain a realizable sequence.

Main theorem – general surface

Theorem (DeVos, Georgakopoulos, Mohar, Š., 2009)

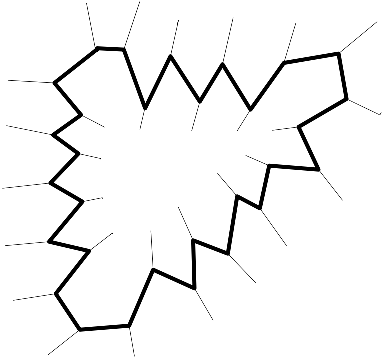
Let $(p_k)_{k \neq 5,7}$ be a finite sequence of non-negative integers, let S be a closed surface, and let w be a positive integer. Then there exist infinitely many pairs of integers p_5 and p_7 such that there is a 3-connected map realizing S, with face-width at least w, having precisely p_k faces of length k.

Triarcs



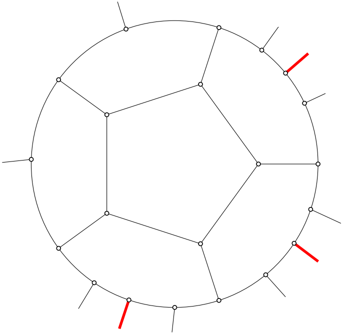
- a useful planar cubic graph

Triarcs



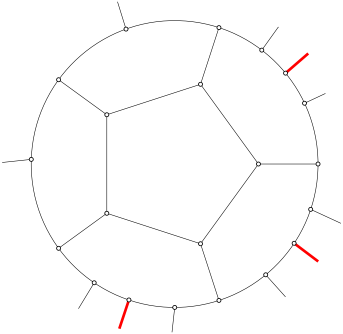
- generalization: planar graph, vertices of degree 3 and 2 (degree 2 are on the boundary, "every other one, plus three").

Basic triarcs



- example: a $(2, 2, 4)$-triarc
- this can be modified by using n-gon instead of pentagon and obtaining any (a, b, c)-triarc with $a + b + c = n + 3$. Such construction, with $a = b$ being even will be called *basic triarc*.

Basic triarcs



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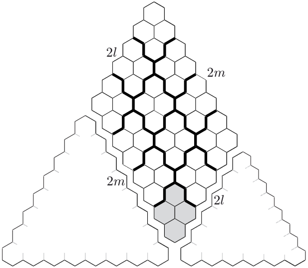
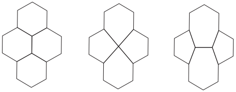
Glueing triarcs together



- two triarcs and several hexagons can be glued to a larger triarc (provided the indicated sides are of even size)
- hexagonal “tiles” can be replaced by ones using only 5-gons and 7-gons:



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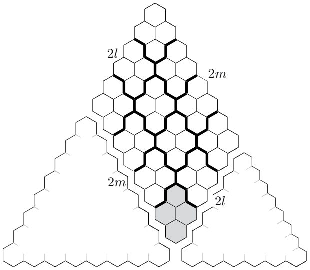

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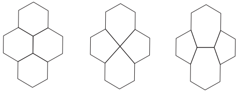

Finishing up

- make basic triarcs and glue them all together
- enlarge the obtained triarcs (by adding triarcs with only 5-gons and 7-gons) to get a (n, n, n)-triarc T with $n = 24k + 8$ (for some k).
- make another (n, n, n)-triarc R using only 5- and 7-gons.
- glue R and T together using the following gadget:

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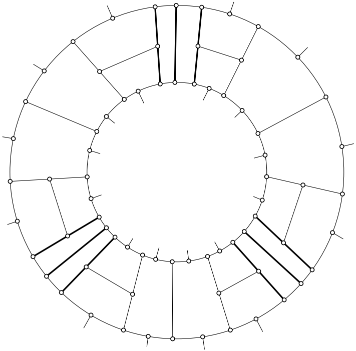
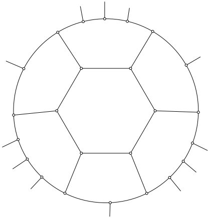
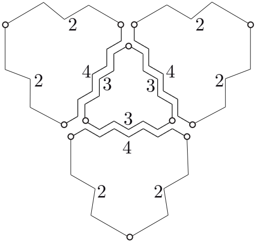


Extending the result: nonplanar graphs

- make two extra triarcs for each handle we need to add (one extra triarc for each crosscap) as follows:





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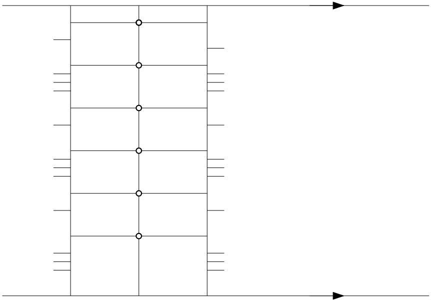

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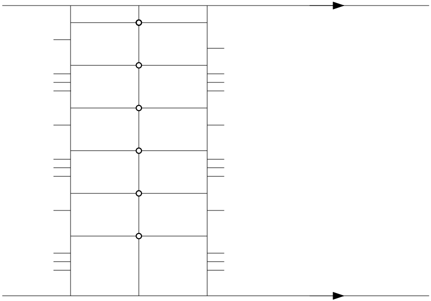

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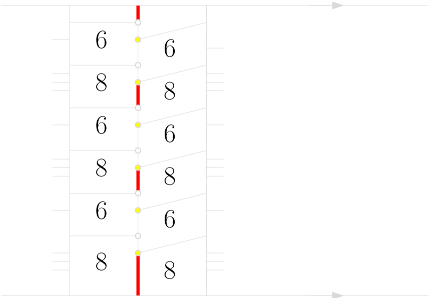

- . . . this doesn’t work either (we can’t add 6-gons, 8-gons)
- contract and uncontract the red edges!

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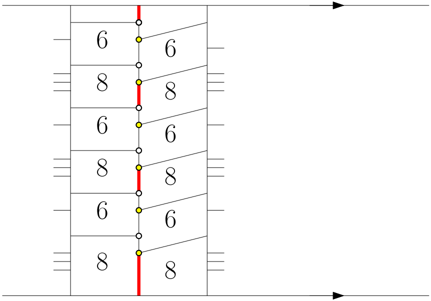

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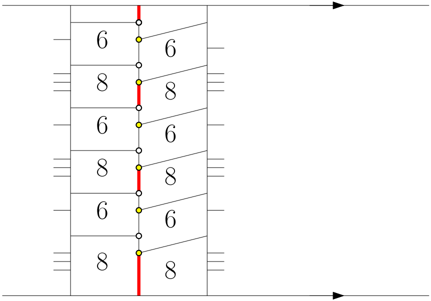

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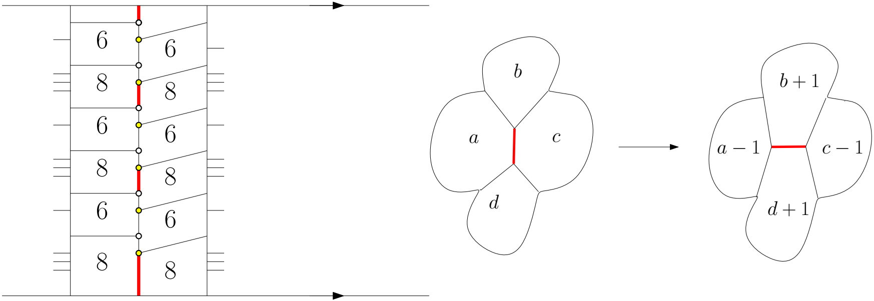

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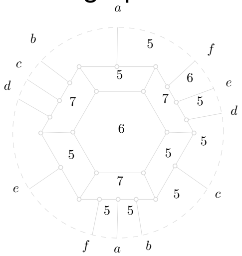

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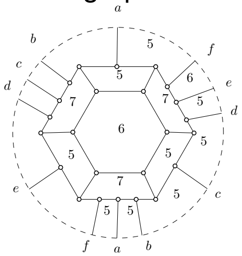

Extending . . . achieving large face-width

- To make a graph with prescribed face-lengths and large face-with, we modify the above construction.
- We do “the same” but with auxiliary $6N$-gons instead of 6-gons. (For N sufficiently large and odd.)

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Future work: Not all is possible

Theorem (Jendrol',Jucovic 1972)

On the torus there is precisely one admissible sequence (namely $p_5 = p_7 = 1$ and $p_i = 0$ for $i \notin \{5, 7\}$), for which an Eberhard-type result with added hexagons does not hold. Explicitly: there is no cubic graph embedded on torus with one pentagon, one heptagon and the rest of hexagons.

Question

Given:

- S — *a closed surface* S
- (p_k) — *satisfies Euler's condition:* $\sum_{k\geq 3}(6-k)p_k = 6\chi(S)$.
- (q_k) — *is neutral:* $\sum_{k\geq 3}(6-k)q_k = 0$.

Is it true (with finitely many exceptions (p, q)*) that*

$(\exists n \in \mathbb{N})$ *such that* $p + nq$ *is realizable in* S*?*

(As mentioned above, if S is the torus then the list of exceptional pairs (p, q) cannot be empty.)