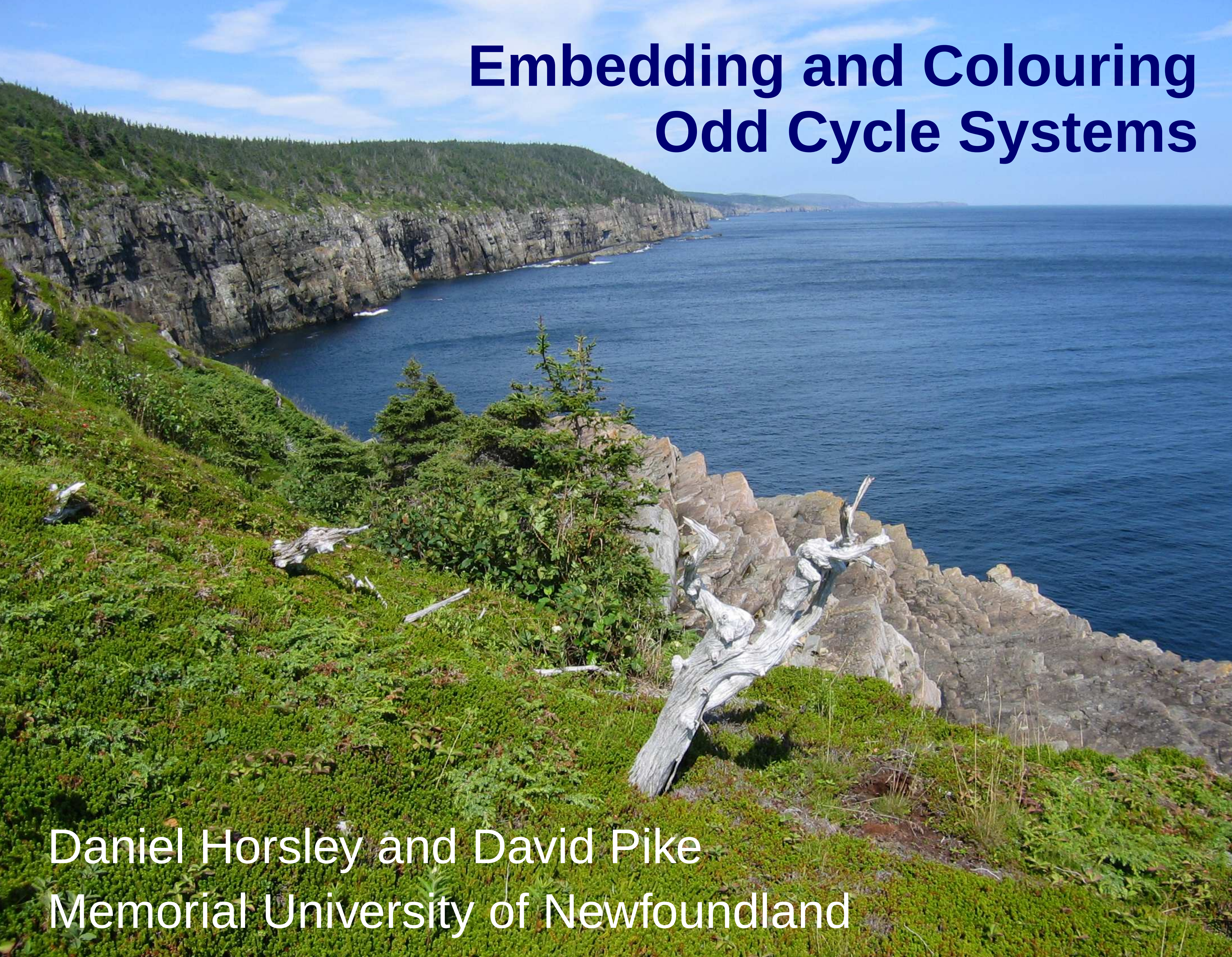

Embedding and Colouring Odd Cycle Systems

Daniel Horsley and David Pike
Memorial University of Newfoundland

Definition:

An m-cycle system of order v consists of a decomposition of K_v into edge-disjoint cycles, each having length m.

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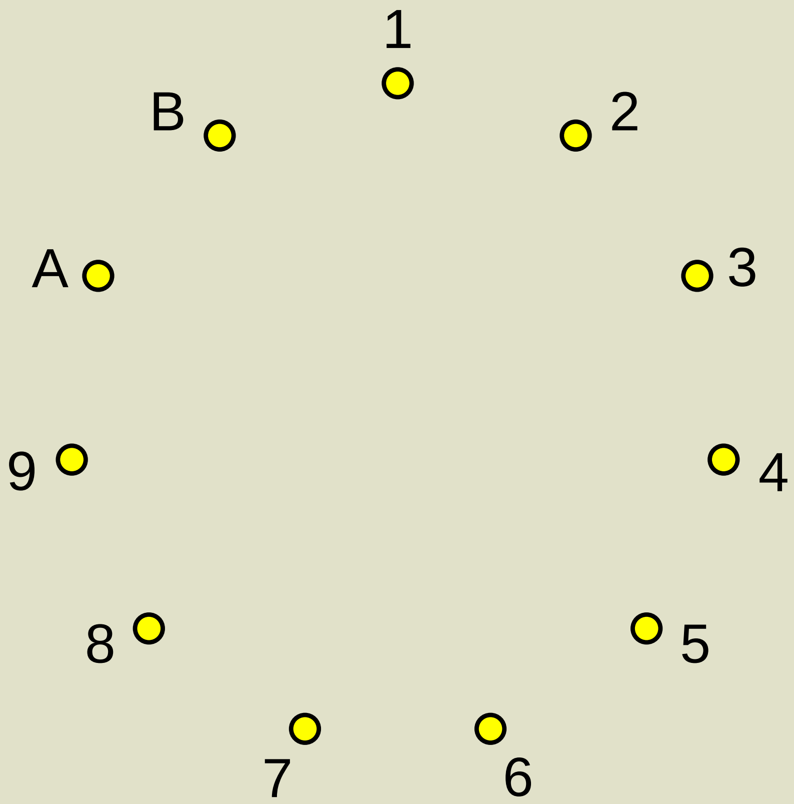
Example: A 5-cycle system of order 11:



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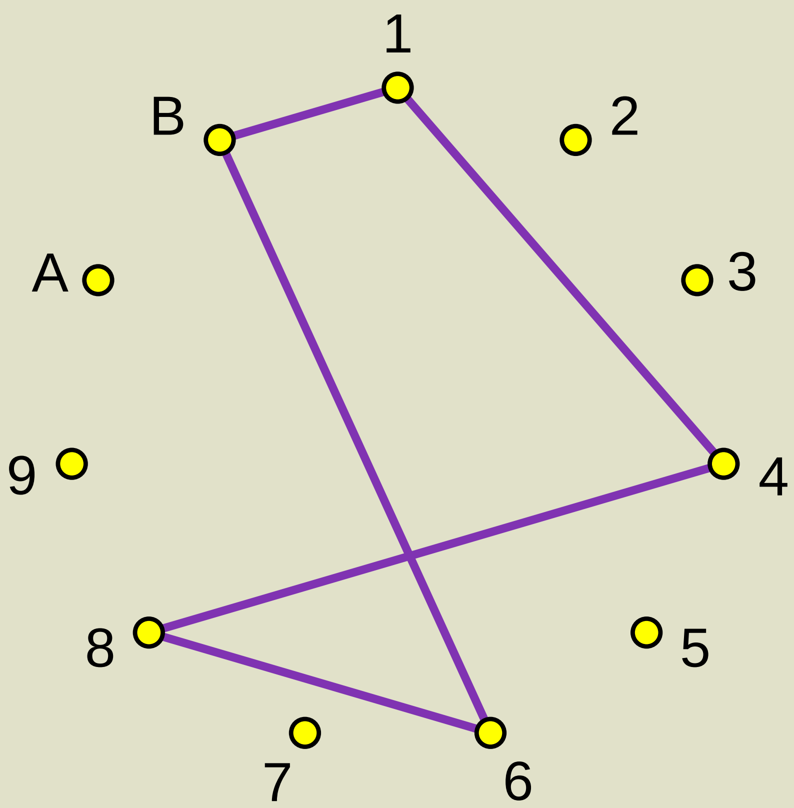

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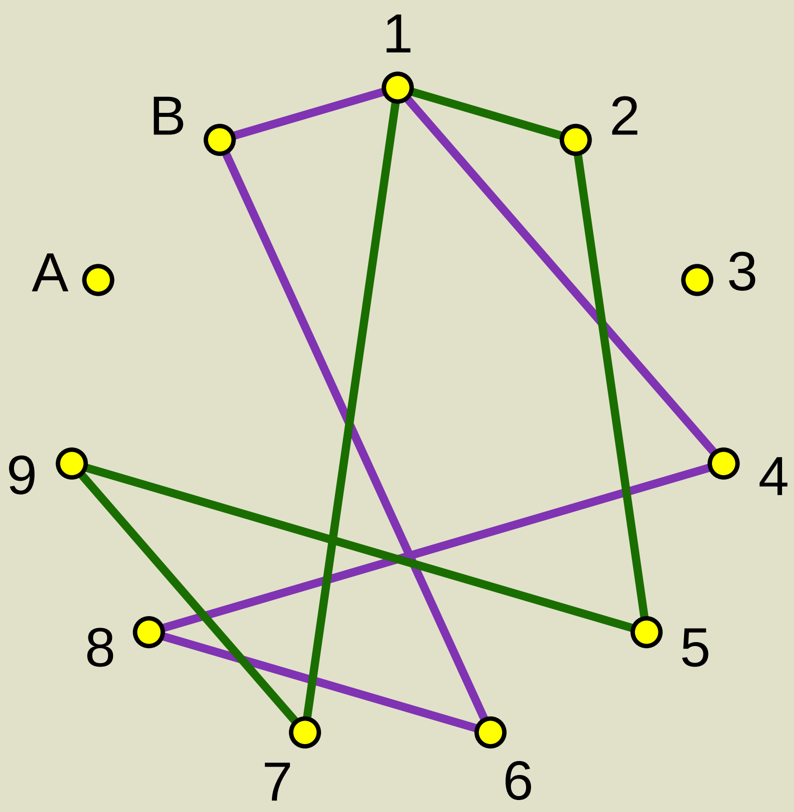

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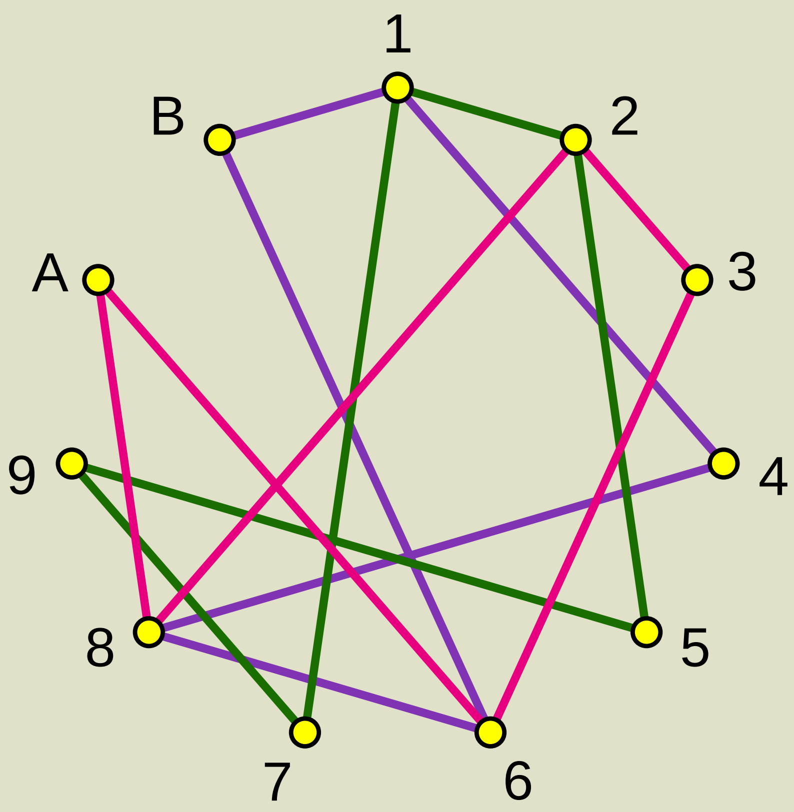

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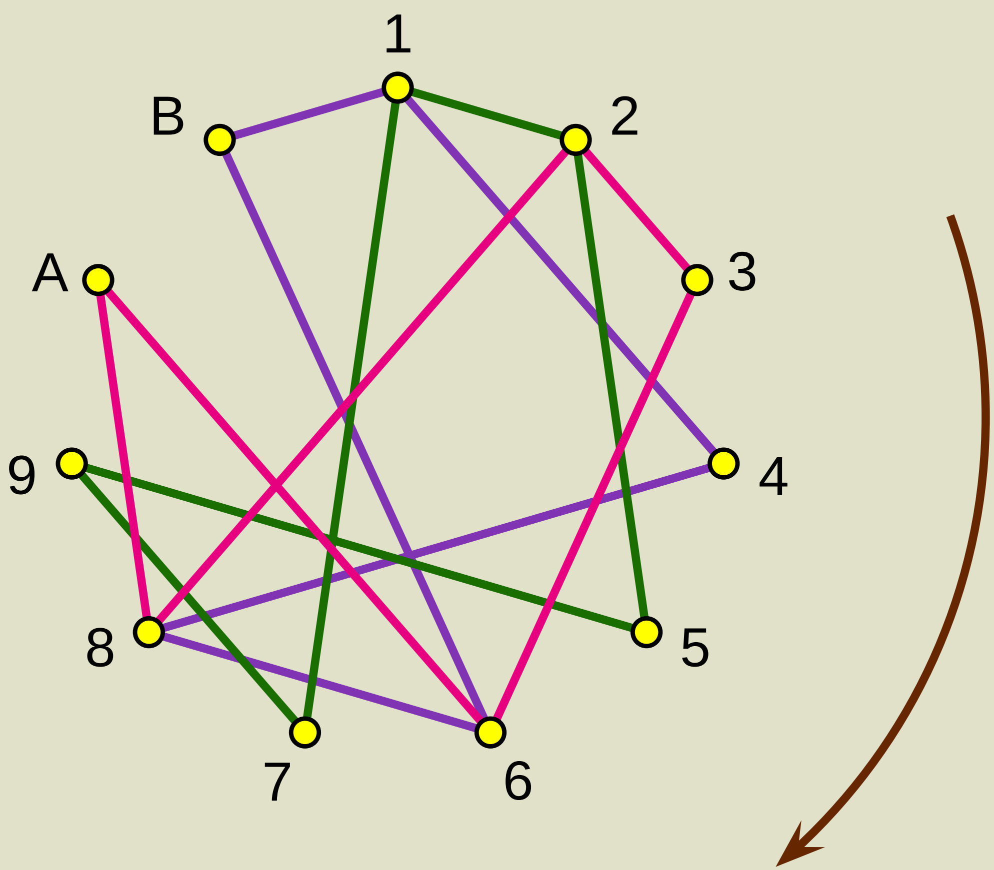

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Theorem (Alspach and Gavlas, 2001; Šajna, 2002):

An m-cycle system of order v exists if and only if v is odd, $v \geqslant m$, and m divides $\binom{v}{2}$.

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Cycle Length m	Admissible Orders v
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7	1 or 7 (mod 14)
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15	1, 15, 21 or 25 (mod 30)

Definition:

A weak k-colouring of a cycle system $\mathcal{S}$ consists of a partition of the vertices of $\mathcal{S}$ into k colour classes such that no cycle of $\mathcal{S}$ is monochromatic.

A cycle system $\mathcal{S}$ is said to be k-chromatic if k is the smallest integer for which $\mathcal{S}$ admits a k-colouring.

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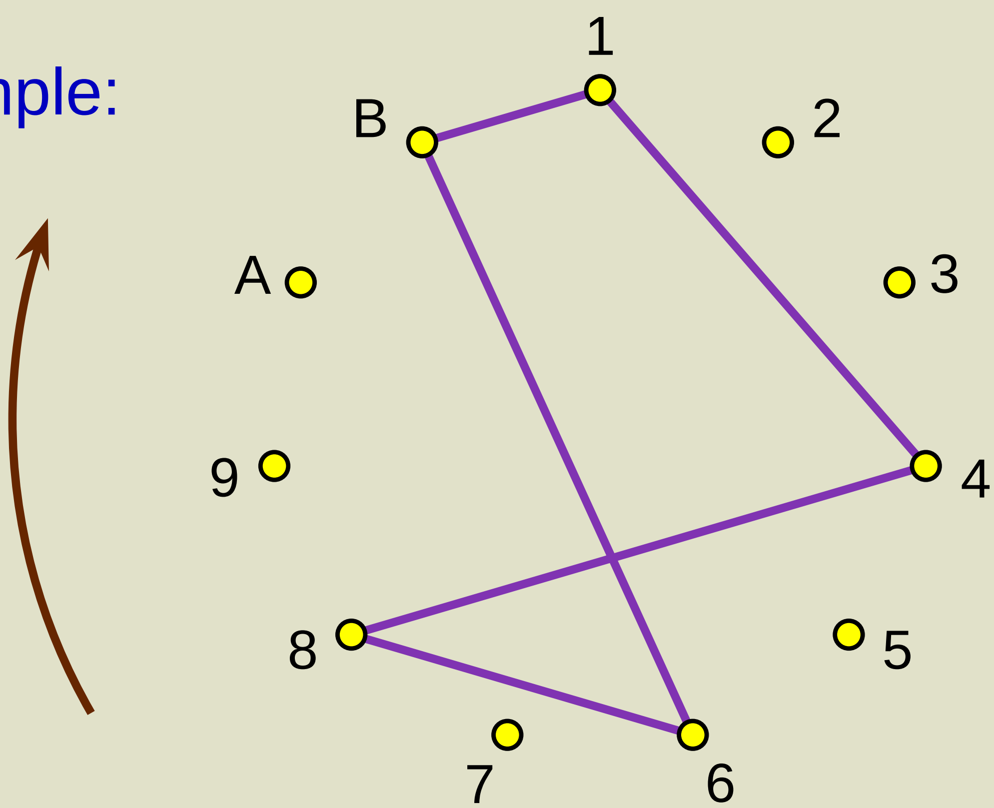

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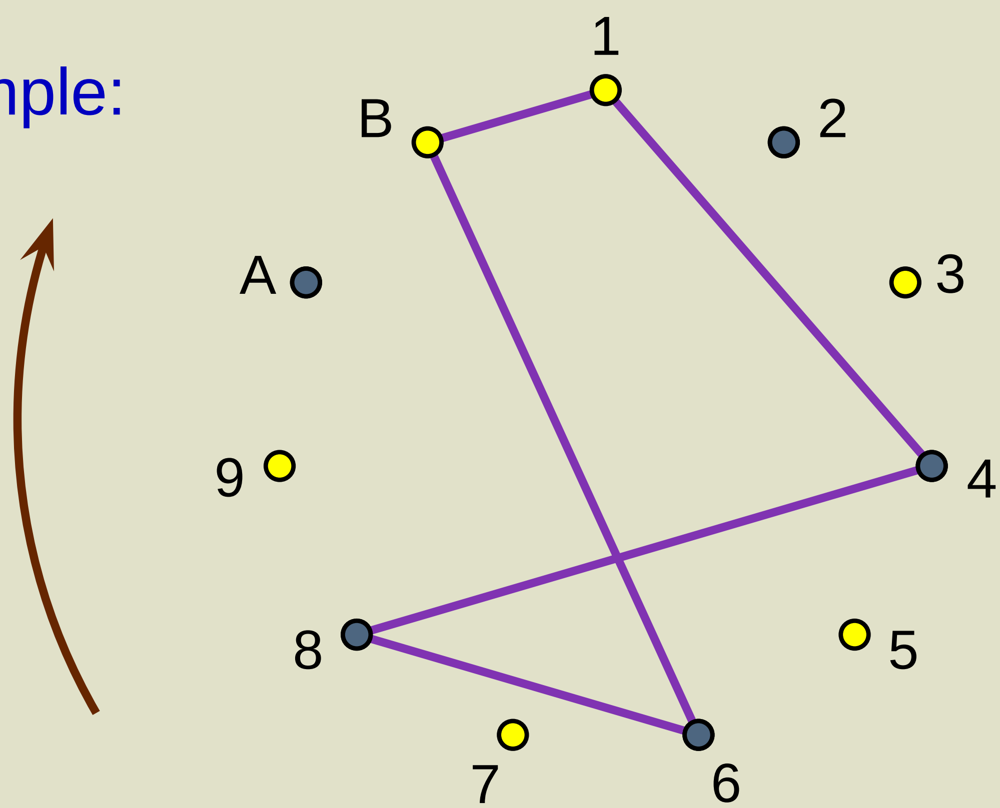

This system is 2-chromatic.

Some History – Triple Systems

- Every STS(v) with $v \geqslant 7$ requires at least 3 colours.
  (Rosa and Pelikán, 1970)
- Every STS(v) with $7 \leqslant v \leqslant 15$ is 3-chromatic.
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- There is a 4-chromatic STS(21). (Haddad, 1999)
- There is a 5-chromatic STS(63). (Fugère, Haddad and Wehlau, 1994)
- There is a 6-chromatic STS(243). (Bruen, Haddad and Wehlau, 1998)

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Definition:

A partial m-cycle system of order u consists of a set of edge-disjoint m-cycles on u vertices.

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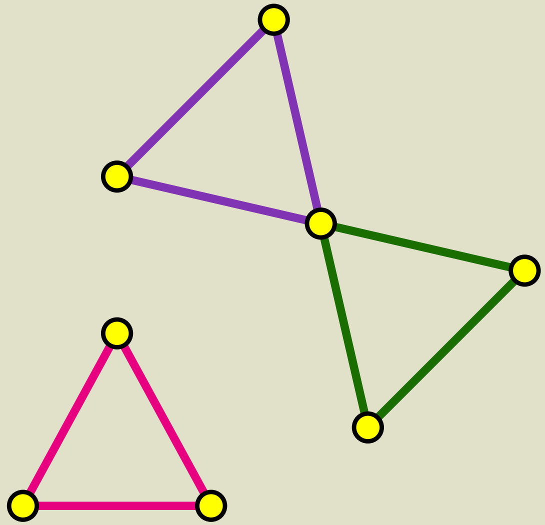

PTS(8)

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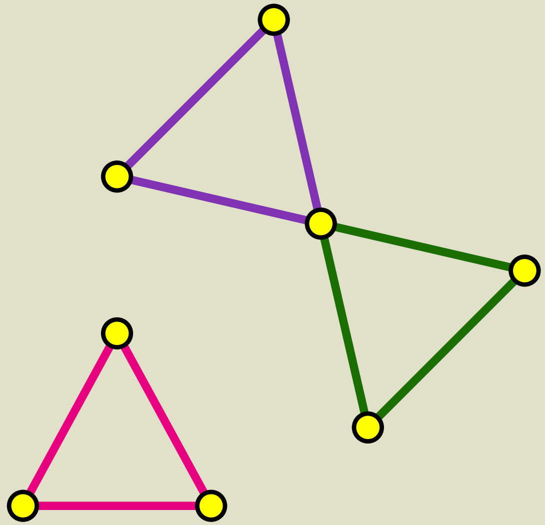

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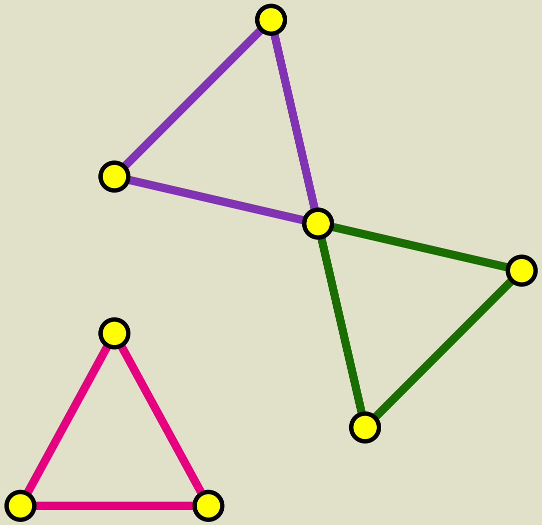

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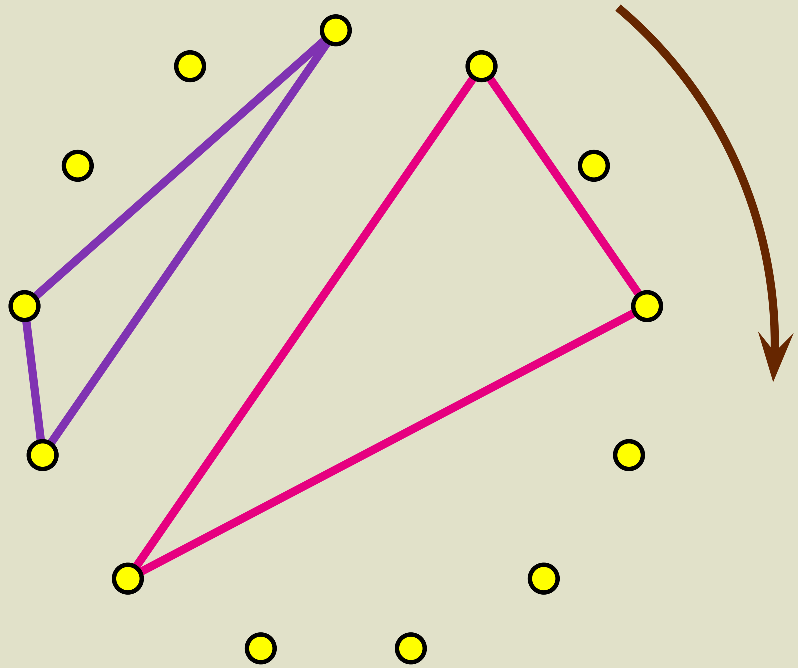

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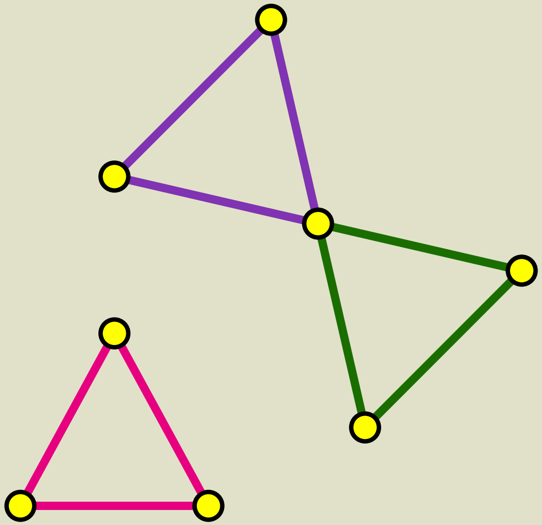

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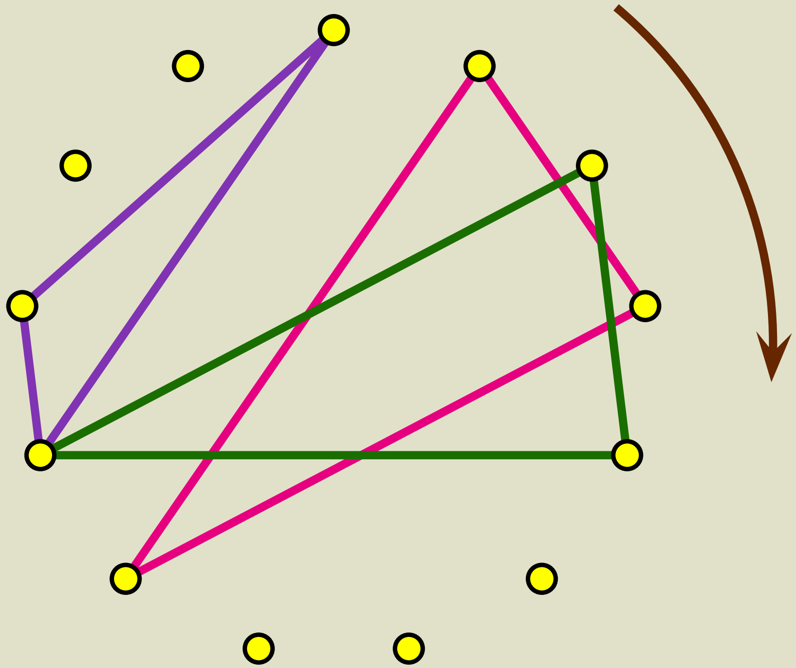

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Theorem (Erdős and Hajnal, 1966; Lovász, 1968):

Let $m \geqslant 2$, k and s be natural numbers.
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We now wish to show that weakly k-chromatic partial m-cycle systems can be embedded into weakly k-chromatic m-cycle systems.

To convey some of the flavour of such embeddings, we focus on $k = 3$ and $m = 5$.

Let $\mathcal{P}$ be a weakly 3-chromatic partial 5-cycle system of some order u, on vertex set $\mathbb{Z}_u$.

Let $\alpha : \mathbb{Z}_u \to \{c_1, c_2, c_3\}$ be a colouring of $\mathbb{Z}_u$ such that no cycle of $\mathcal{P}$ is monochromatic under α.

Let $v \geqslant 10u + 7$ be 5-admissible (so $v \equiv$ 1 or 5 (mod 10)).
Let $t, w \in \mathbb{Z}$ such that $v = (2t + 1)(5) + w$ and $2 \leqslant w \leqslant 11$.
Necessarily w is even and $t \geqslant u$.

We will embed $\mathcal{P}$ in a 3-chromatic 5-cycle system $\mathcal{S}$ of order v, on vertex set $(\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

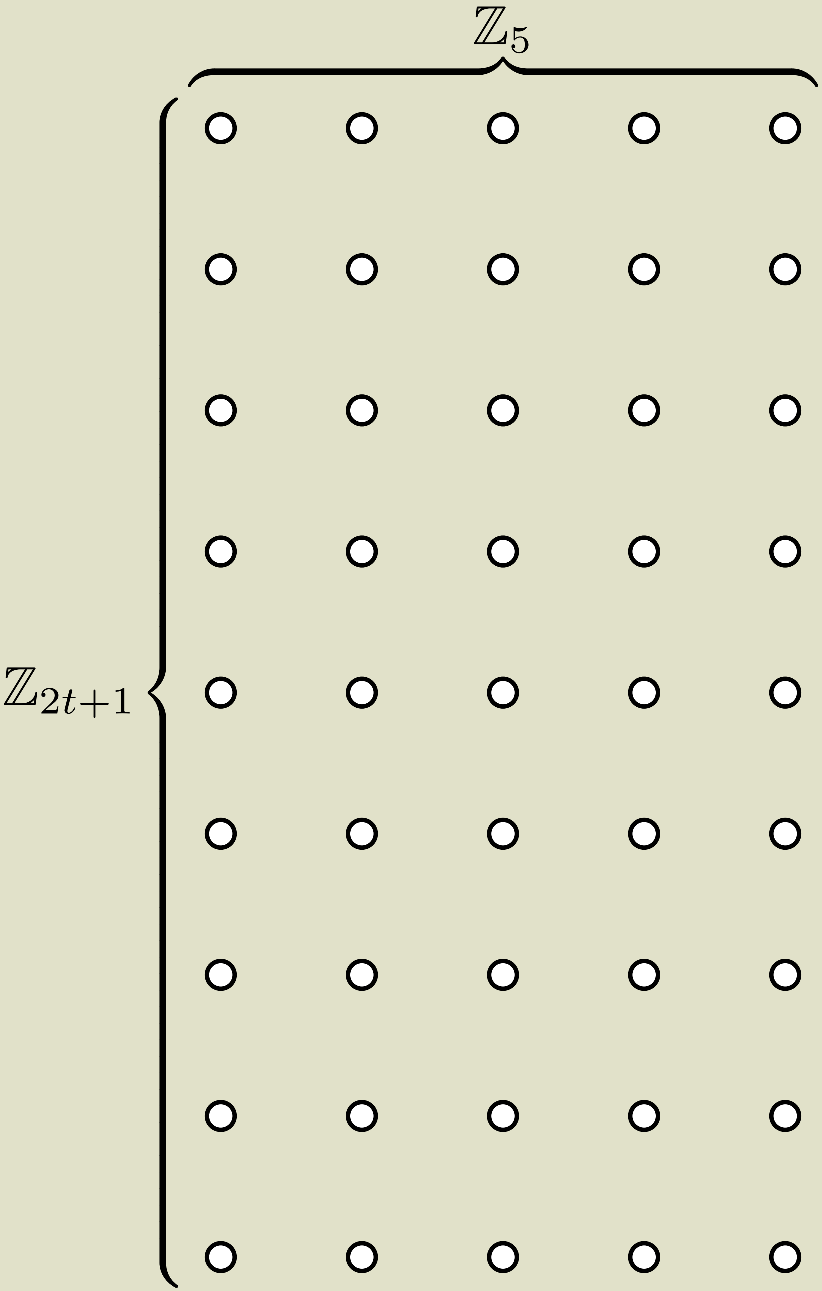
$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Let β be a colouring of V such that:

- $\beta(x) = c_1$ for all $x \in W$
- $\beta(x) = c_2$ for all $x \in W'$
- $$\beta\big((x,i)\big) = \begin{cases} \alpha(x) & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{0,2\} \\ \pi(\alpha(x)) & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{1,3\} \\ \pi^2(\alpha(x)) & \text{if } x \in \mathbb{Z}_u \text{ and } i = 4 \\ c_1 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{0,2\} \\ c_2 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{1,3\} \\ c_3 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i = 4 \end{cases}$$

  where π is the permutation (c_1, c_2, c_3).

$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.



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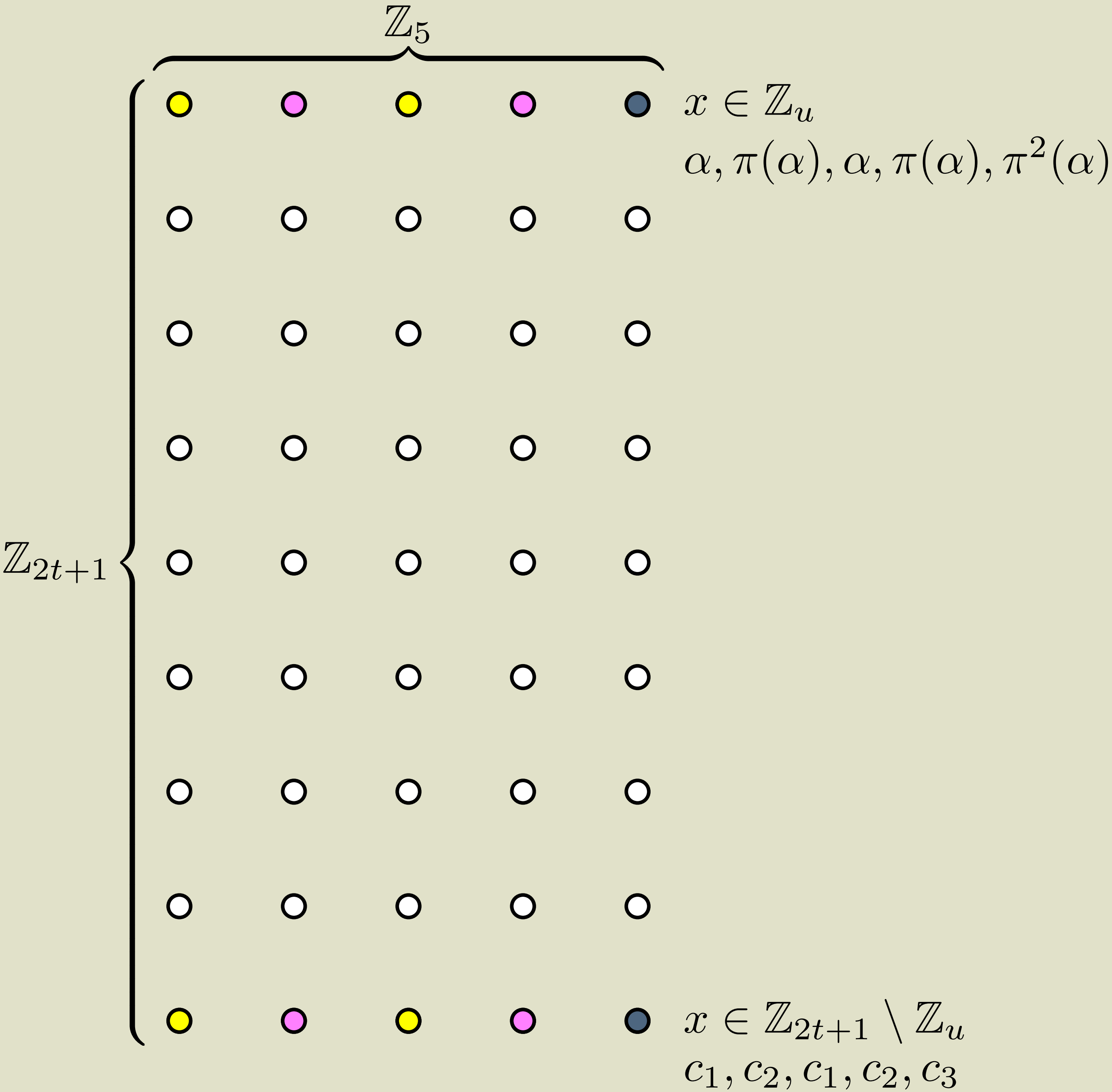

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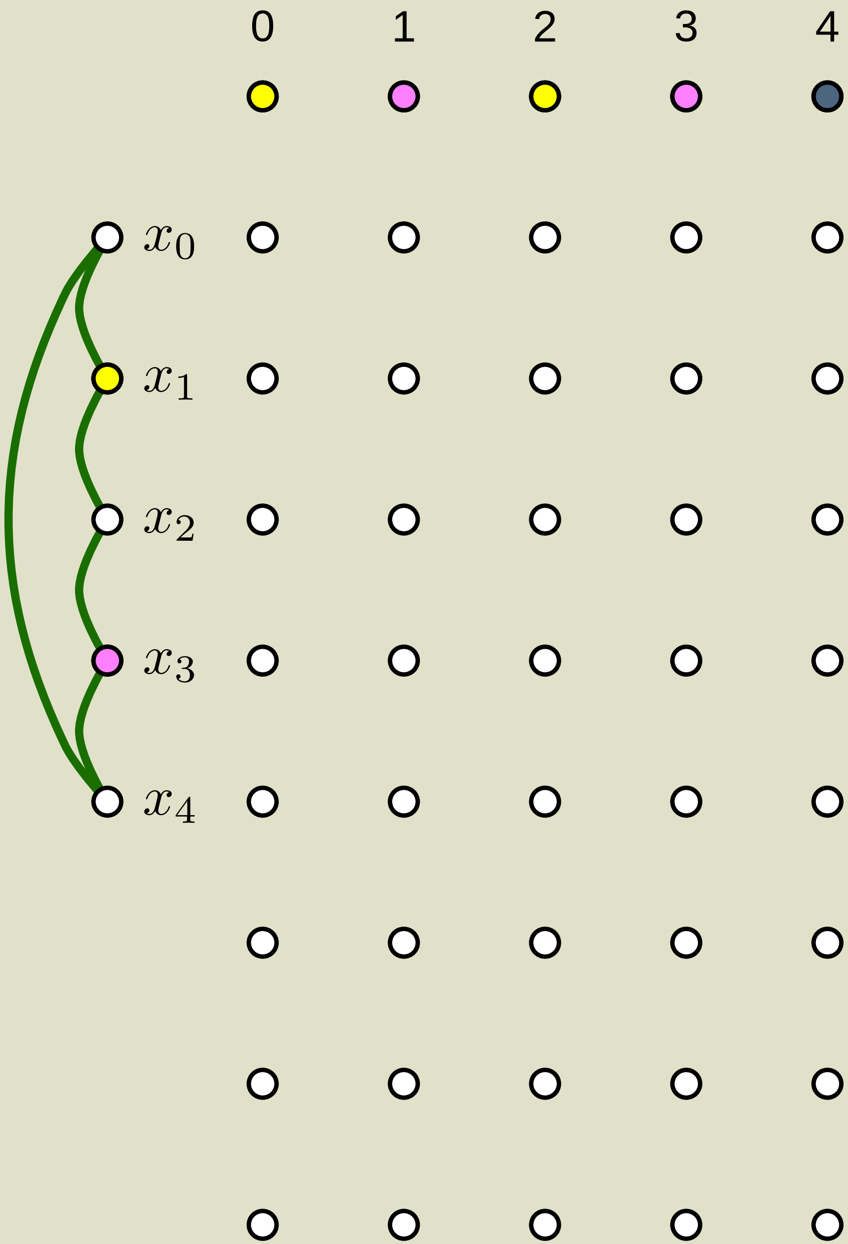
Step 1 of 4



$x \in \mathbb{Z}_u$

$\alpha, \pi(\alpha), \alpha, \pi(\alpha), \pi^2(\alpha)$

For each $C = (x_0, x_1, x_2, x_3, x_4)$ in $\mathcal{P}$ there are vertices x_1 and x_3 with different colours.

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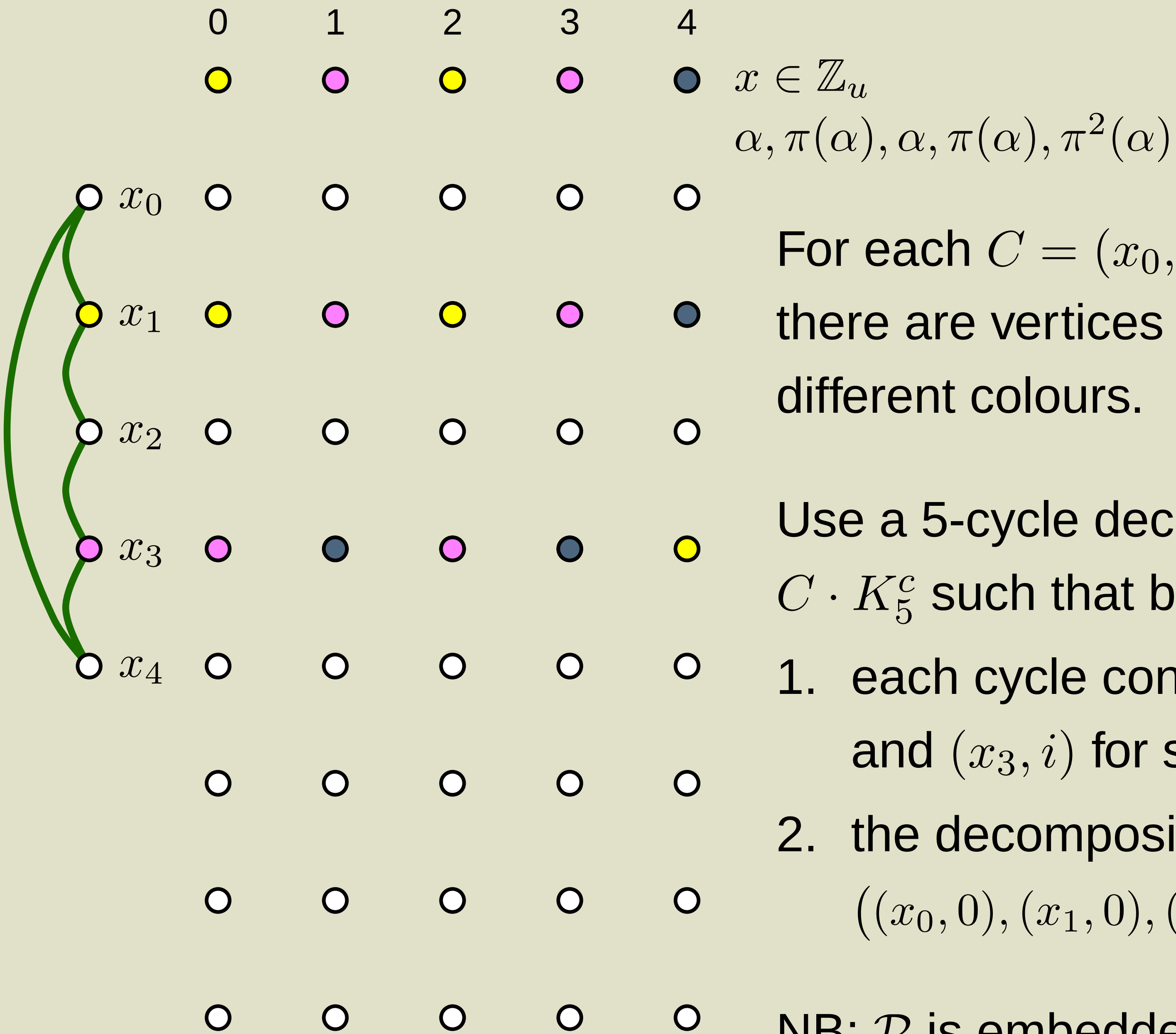


Step 1 of 4

For each $C = (x_0, x_1, x_2, x_3, x_4)$ in $\mathcal{P}$ there are vertices x_1 and x_3 with different colours.

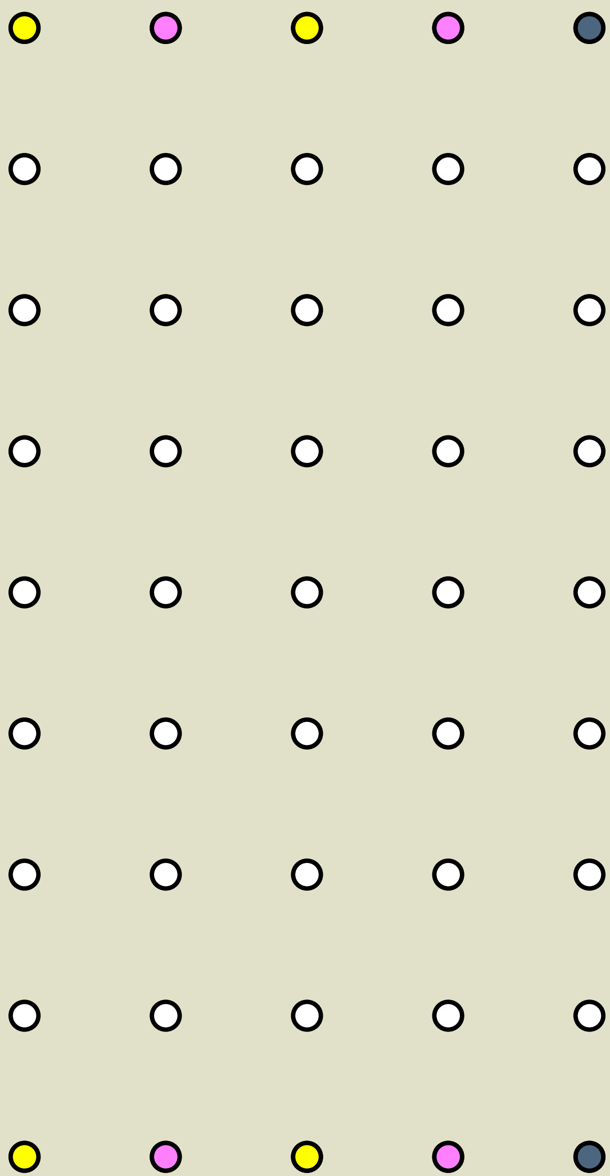
Use a 5-cycle decomposition of $C \cdot K_5^c$ such that both:

1. each cycle contains both (x_1, i) and (x_3, i) for some i
2. the decomposition includes $\big((x_0, 0), (x_1, 0), (x_2, 0), (x_3, 0), (x_4, 0)\big)$

NB: $\mathcal{P}$ is embedded within $\mathbb{Z}_u \times \{0\}$.

$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Step 2 of 4



$x \in \mathbb{Z}_u$

$\alpha, \pi(\alpha), \alpha, \pi(\alpha), \pi^2(\alpha)$

Let G be K_{2t+1} with the edges of $\mathcal{P}$ removed.

$x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u$

c_1, c_2, c_1, c_2, c_3

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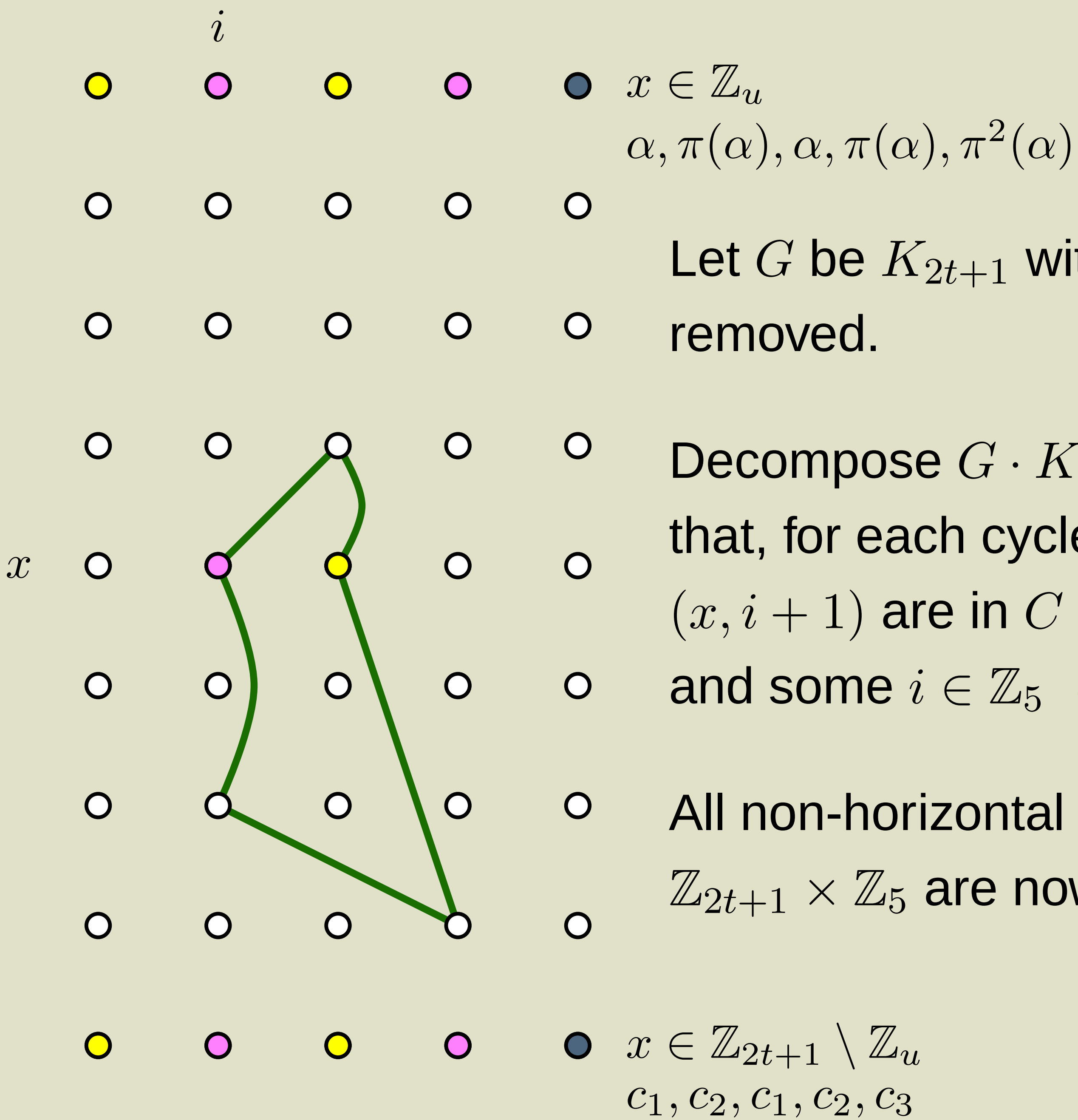


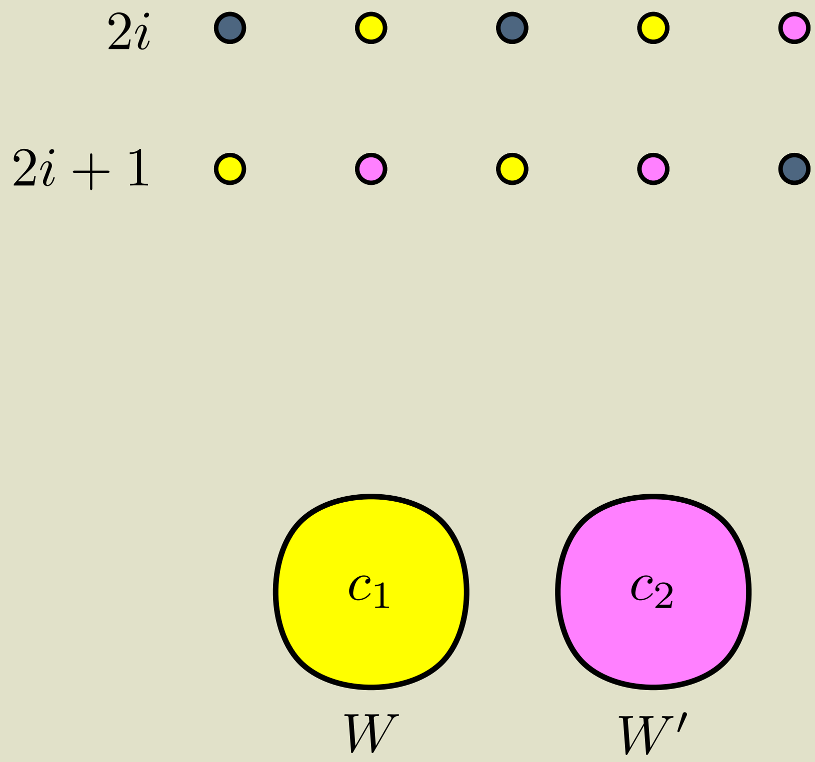
Let G be K_{2t+1} with the edges of $\mathcal{P}$ removed.

Decompose $G \cdot K_5^c$ into 5-cycles such that, for each cycle C, (x, i) and $(x, i+1)$ are in C for some $x \in \mathbb{Z}_{2t+1}$ and some $i \in \mathbb{Z}_5$ (Lindner and Rodger, 1993)

All non-horizontal edges of $\mathbb{Z}_{2t+1} \times \mathbb{Z}_5$ are now used.

$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Step 3 of 4



Let G be $K_w^c \vee (K_5 \cup K_5)$, formed on $W \cup W' \cup (\{2i, 2i+1\} \times \mathbb{Z}_5)$.

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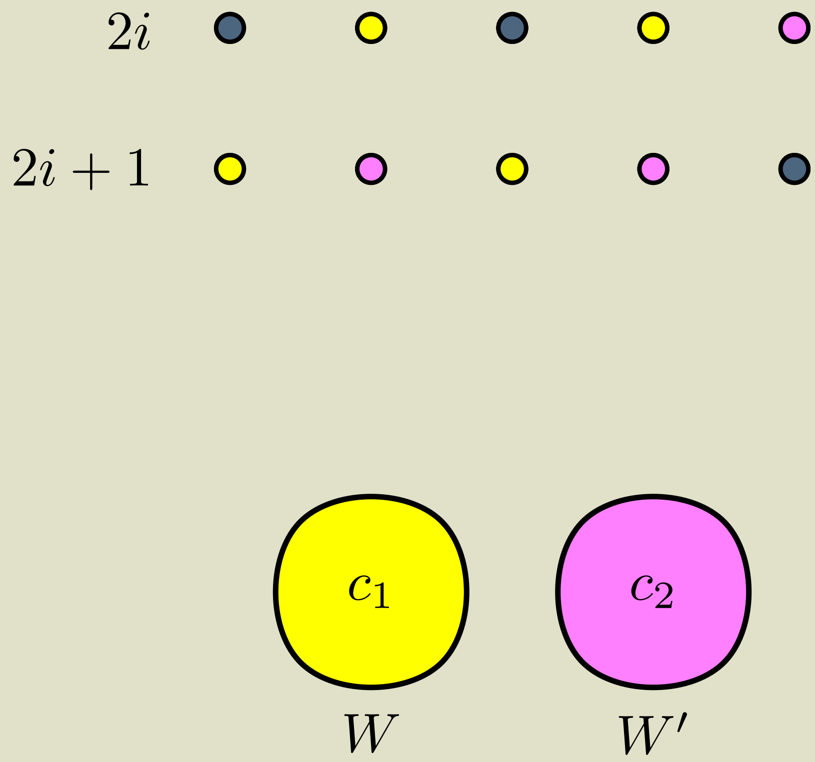

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Decompose G into 5-cycles such that, for each cycle C, either

1. $V(C) \cap W \neq \emptyset$ and $V(C) \cap W' \neq \emptyset$

   or

2. $V(C) \cap (W \cup W') = \emptyset$

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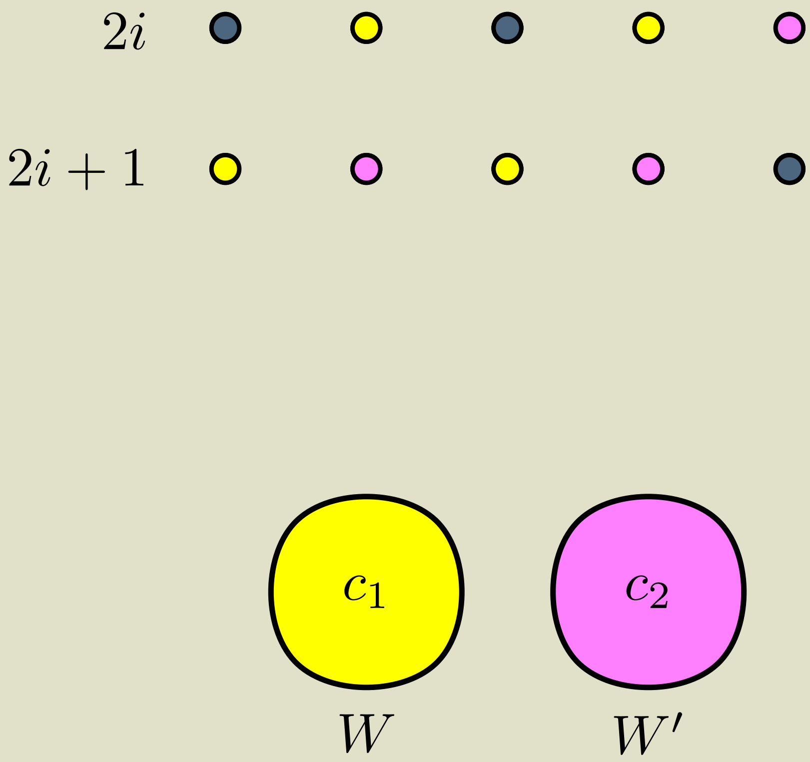
Step 3 of 4



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1. $V(C) \cap W \neq \emptyset$ and $V(C) \cap W' \neq \emptyset$ or
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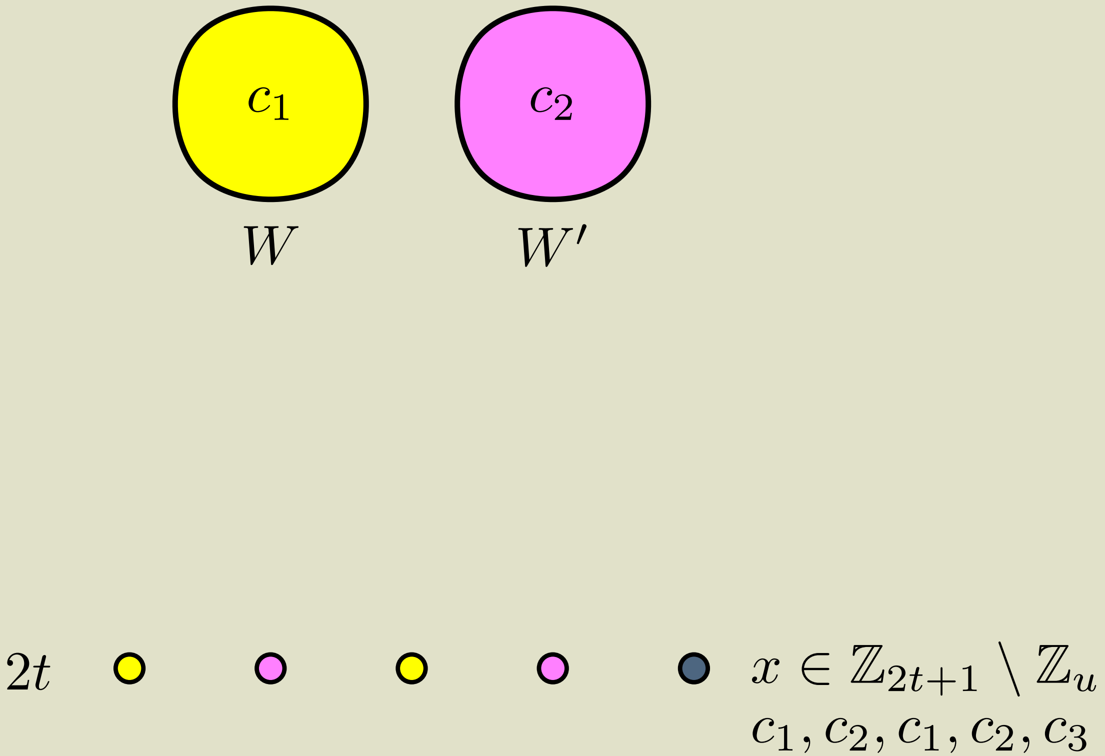
Do this for $i = 0, \ldots, t-1$

$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Step 4 of 4

Let G be K_{w+5} on $W \cup W' \cup (\{2t\} \times \mathbb{Z}_5)$.



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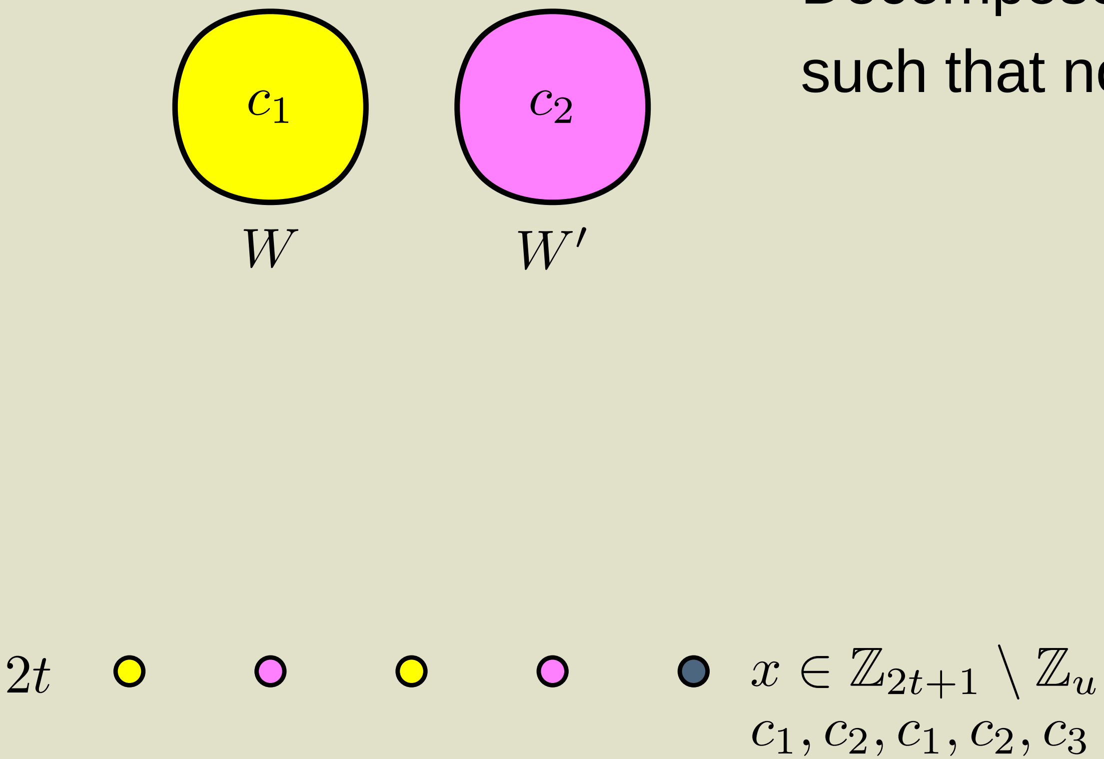
Step 4 of 4

Let G be K_{w+5} on $W \cup W' \cup (\{2t\} \times \mathbb{Z}_5)$.

Decompose G into 5-cycles such that no cycle is monochromatic.

QED



Theorem (Horsley and Pike):

Let k and m be integers such that $k \geqslant 2$, $m \geqslant 3$ and $(k, m) \neq (2, 3)$. Then there is an integer $n'_{k,m}$ such that there exists a weakly k-chromatic m-cycle system of order v for all m-admissible integers $v \geqslant n'_{k,m}$.

Theorem (Horsley and Pike):

Let $u_{k,m}$ be the minimum order of a weakly k-chromatic partial m-cycle system.

Let $n_{k,m}$ be the smallest m-admissible integer such that there exists a weakly k-chromatic m-cycle system of order v for all m-admissible integers $v \geqslant n_{k,m}$.

Then $u_{k,m} \leqslant n_{k,m} \leqslant 2m(u_{k,m} + 1) + 1$.

Questions for Further Study:

- What about other types of designs?
- Perhaps a chromatic version of Wilson's Theorem?

Acknowledgements:

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