Epidemics in Communication Networks *

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Small World Networks

- significant *clustering* or *locality* (neighbouring nodes have many common neighbours)
- small *diameter* or average distance between nodes

Examples

- Erdös numbers
- world wide web
- spread of contagious diseases
- neural network of flatworm
- electric power grids

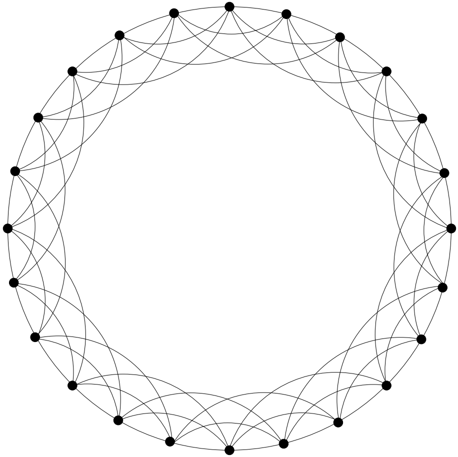
Probabilistic Model (Watts-Strogatz)

Consider each edge of a network and move one of its endpoints with probability p.

- $p = 0 \Rightarrow$ structured graphs
- $p = 1 \Rightarrow$ random graphs
- $p \approx .01 \Rightarrow$ small world networks

  distance 20% of original graph
  clustering 95% of original graph

Initial Graph for Probabilistic Model



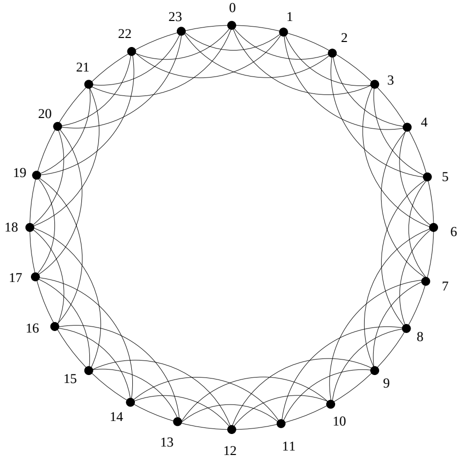
Notation

- *network* is $G = (V, E)$ of *order* $n = |V|$
- *maximum degree* is Δ
- *diameter* is D_G or D
- *clustering* is a measure of the connectedness of a graph (definition later)

Circulant Graphs

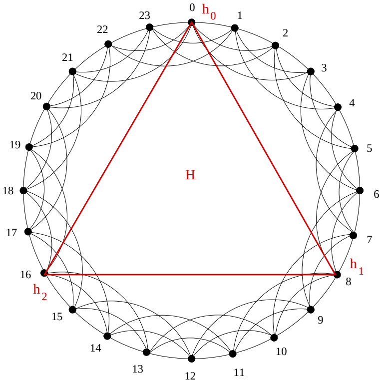
- $C_{n,\Delta}$, Δ even, is the circulant graph $C(n; 1, 2, \cdots, \frac{\Delta}{2})$
- n nodes labelled with integers modulo n
- Δ links per node, each node i adjacent to $i \pm 1, i \pm 2, \cdots, i \pm \frac{\Delta}{2} \pmod{n}$.
- $C_{n,\Delta}$ has diameter $D_{C_{n,\Delta}} = \lceil \frac{n-1}{\Delta} \rceil$.

Example



Circulant graph $C_{n,\Delta}$ with $n = 24$ and $\Delta = 6$

Deterministic Model



Choose h equally spaced nodes of $C_{n,\Delta}$ to be *hubs*.

Interconnect the hubs using a second graph H to obtain $C_{n,\Delta,h}$.

Subgraph of circulant between two hubs is a *segment*.

Diameter of Deterministic Model

Theorem The diameter of $C_{n,\Delta}$ can be reduced to D using a hub graph H with approximately $h = \frac{2n}{\Delta(D - D_H)}$ nodes.

Theorem When a graph H with h nodes and diameter D_H is used to interconnect h hubs of $C_{n,\Delta}$, the diameter is approximately $\frac{2n}{\Delta h} + D_H$. ($\frac{n}{\Delta h}$ is the maximum distance from any node of $C_{n,\Delta}$ to the nearest hub.)

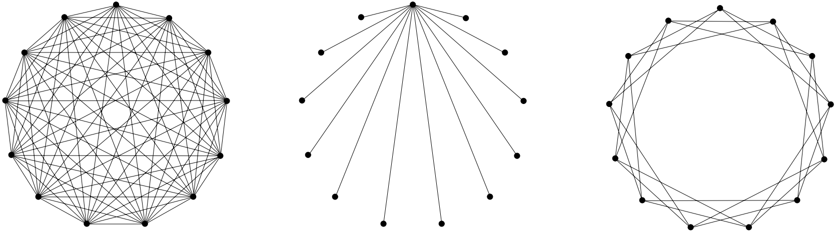
Examples Consider $C_{n,\Delta}$ with $n = 1000$, $\Delta = 10$, $D_{C_{n,\Delta}} = 100$.

- To reduce the diameter to 34 requires 7 hubs (first theorem).
- With $h = 4$ hubs, the diameter is $50 + D_H$ (second theorem).

Hub Graphs

- K_h (complete graph): diameter 1 but increases degree of hubs by $h-1$
- $K_{1,h-1}$ (star graph): diameter 2, adds minimum number of edges, one hub has high degree
- $C(n; a, b)$ (double-step graph): circulant graph with each node i adjacent to $i \pm a, i \pm b \pmod{n}$, diameter is $D = \left\lceil \frac{-1+\sqrt{2n-1}}{2} \right\rceil$ when $a = D$ and $b = D + 1$, increases degree of hubs by 4

Examples of Hub Graphs



Complete, star, and double-step hub graphs with $h = 13$

Examples of Diameter with Hub Graphs

Consider $C_{n,\Delta}$ with $n = 10,000$, $\Delta = 10$, and $D_{C_{n,\Delta}} = 1000$. Using $h = 50$ hubs:

- K_{50} – diameter 41, degree of hubs is 59
- $C(50; 5, 6)$ – diameter 45, degree of hubs is 14

50 hubs is 0.5% of n but diameter decreases by factor of nearly 25
20 hubs are needed to decrease diameter by factor of 10

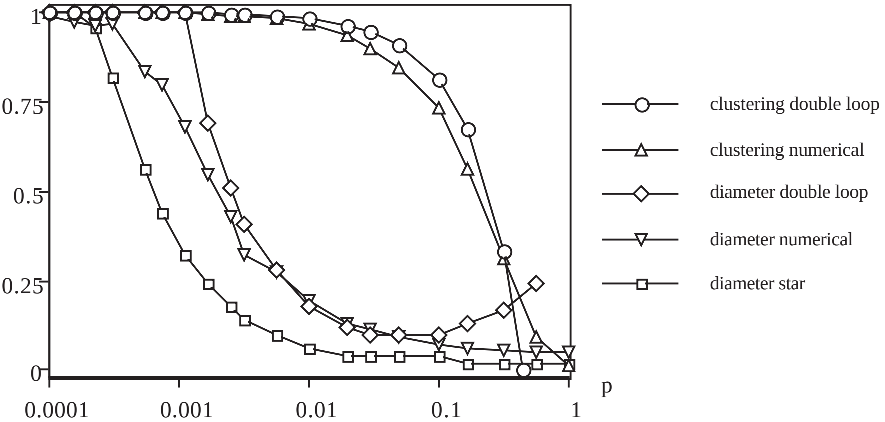
Clustering

For each node i of a graph G, let n_i be the number of neighbours of i. Let $\mathcal{C}_i$ be the fraction of the $\frac{n_i(n_i-1)}{2}$ possible edges among the neighbours of i that are present in G. The *clustering parameter* of G, $\mathcal{C}_G$, is the average over all nodes i of $\mathcal{C}_i$.

Theorem The clustering parameter of $C_{n,\Delta}$ is

$$\mathcal{C}_{C_{n,\Delta}} = \frac{3(\Delta-2)}{4(\Delta-1)} \approx \frac{3}{4}.$$

Comparison of Probabilistic and Deterministic Models



Clustering and diameter for graphs based on $C_{1000,10}$

Epidemics

- Flu epidemic - spreads to many people simultaneously but individuals remain contagious for a short time
- Aids epidemic - spreads to one person at a time but individuals remain contagious for a long time

How can we model the infectiousness of a disease and how long an individual is contagious?

Broadcasting

Infectious diseases spread in a broadcast-like pattern.

- at time $t = 0$, an individual (originator) starts to spread the disease
- an individual (node) remains *active* (*contagious*) for A time units (*rounds*)
- an active node can infect $k \leq \Delta$ neighbours during each round
- an active node becomes permanently *inactive* after A rounds

Special Cases

- $A = \infty$ is normal broadcasting with a *k-port model*.
- $A = 1$ has been studied with probabilistic models.
- We will study $A = 1$ with a deterministic model.

Definitions

$\mathcal{T}_{k,\mathcal{A}}(u)$: *broadcast time* for originating node u when

- a node can infect $k \leq \Delta$ of its neighbours during each active time step
- a node remains *active* for $\mathcal{A}$ rounds after infection

The *broadcast time of graph G* is
$\mathcal{T}_{k,\mathcal{A}}(G) = \max\{\mathcal{T}_{k,\mathcal{A}}(u) | u \in V(G)\}$.
Simplify to $\mathcal{T}_k(u)$ and $\mathcal{T}_k(G)$ since only $A = 1$ is considered.

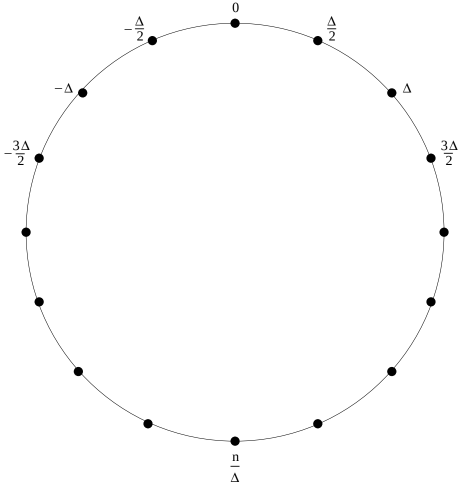
Broadcast Time in $C_{n,\Delta}$

Lower Bound Diameter of $C_{n,\Delta}$, Δ even, is $\lceil \frac{n-1}{\Delta} \rceil$.

Theorem For $n \geq 2\Delta(\lceil \log_k \rceil - 1) + 2$

- $\mathcal{T}_k(C_{n,\Delta}) = n - 1$ if $k = 1$
- $\mathcal{T}_k(C_{n,\Delta}) = \lceil \frac{n-1}{\Delta} \rceil$ if $2 \leq k \leq \Delta$

Sections of a Circulant

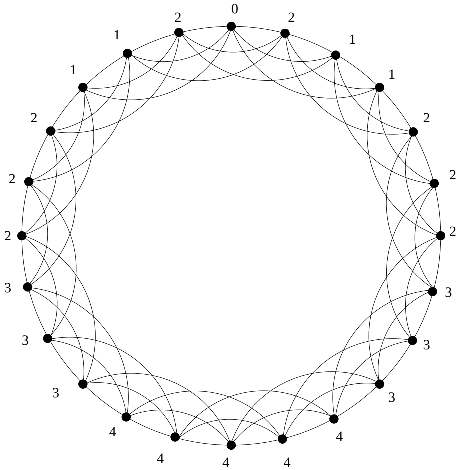


Sections of $C_{n,\Delta}$

Proof Outline

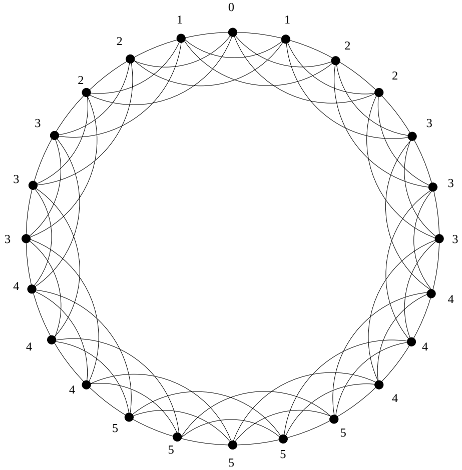
- $k = 1$: one node active during each round $\implies$ $n - 1$ rounds
- $k = \Delta$: spreading pattern is flooding, one section on each side of the originator per round
- $\frac{\Delta}{2} < k < \Delta$:
  - Round 1: originator infects farthest $\frac{k}{2}$ neighbours on each side
  - Round 2: infect remaining nodes in section on each side of originator and more than half of nodes in next section on each side
  - continue until round $\left\lceil \frac{n-1}{\Delta} \right\rceil$
- $2 \leq k \leq \frac{\Delta}{2}$: infect nodes in section on each side of originator in $\lceil \log_k \Delta \rceil$ rounds and then flood for $\left\lceil \frac{n-1}{\Delta} \right\rceil - 1$ more rounds to infect remaining sections giving $\mathcal{T}_k(C_{n,\Delta}) \leq \lceil \log_k \Delta \rceil + \left\lceil \frac{n-1}{\Delta} \right\rceil - 1$

Example for $\frac{\Delta}{2} < k < \Delta$



Optimal broadcast in $C_{n,\Delta}$ for $n = 24$, $\Delta = 6$, $k = 4$

Example for $2 \leq k \leq \frac{\Delta}{2}$

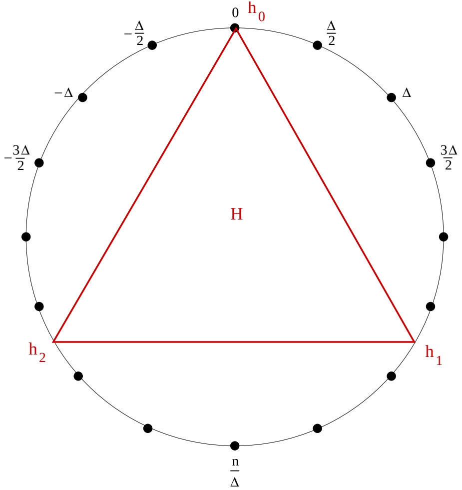


Broadcast in $C_{n,\Delta}$ for $n = 24$, $\Delta = 6$, $k = 2$, $\mathcal{A} = 1$,

Proof Outline for $2 \leq k < \Delta$

- Round 1: originator infects farthest $\frac{k}{2}$ neighbours in section on each side
- Round 2: infect the farthest $\frac{k^2}{2}$ nodes in next section on each side
- Continue to round $\alpha - 1$ where $\alpha = \lceil \log_k \Delta \rceil$
- Round α: infect all nodes in section α and some nodes in section $\alpha - 2$
- Round $\alpha + 1$: infect remaining nodes in section $\alpha - 1$, all nodes in section $\alpha + 1$, and some nodes in section $\alpha - 3$
- Continue flooding in direction away from originator and filling in toward originator until round $2\alpha - 1$

Segments of $C_{n,\Delta,h}$



Segments of $C_{n,\Delta,h}$

Broadcasting in $C_{n,\Delta,h}$

Theorem $T_k(C_{n,\Delta,h}) \approx \frac{2n}{h\Delta} + D_H$ where

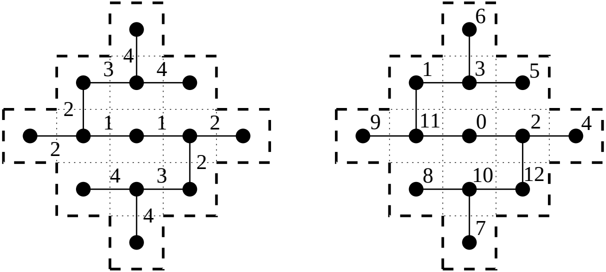
- $H = C(h; a, b)$ (double-step graph)
- $3 \leq k \leq \frac{\Delta}{2}$
- $\Delta \geq 4$, Δ even

Proof Outline

- First phase: originator sends message to nearest hub using chords of length $\frac{\Delta}{2}$ (approximately $\frac{n}{h\Delta}$ rounds worst case)
- Second phase: the informed hub broadcasts in the hub graph H ($D_H = \left\lceil \frac{-1+\sqrt{2h-1}}{2} \right\rceil$ rounds)
- Third phase: each hub broadcasts to the two segments on either side of it (approximately $\frac{n}{h\Delta}$ rounds)

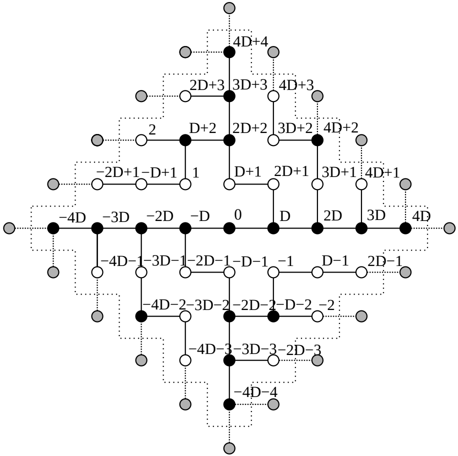
Theorem $T_k(C_{n,\Delta,h}) \approx \frac{3n}{h\Delta} + D_H$, $k = 2$

Broadcast in Tile



Broadcast in a tile with $k = 2$

Broadcast in Modified Tile



Modified broadcast in a tile with $k = 2$

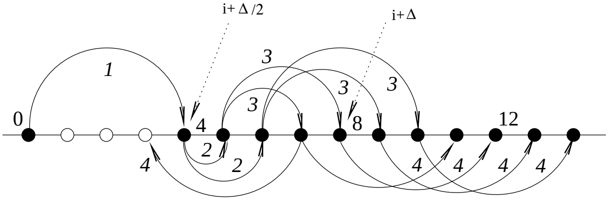
Stopping the Infection

Theorem If no communications are lost, then all nodes of $C_{n,\Delta}$ will be infected for any $2 \leq k \leq \Delta$.

Proof Outline

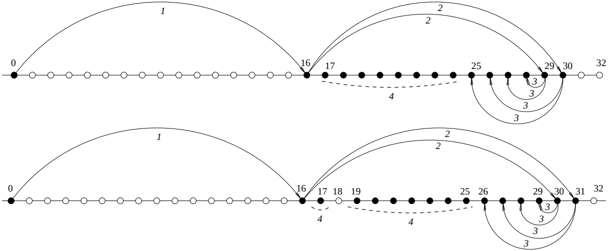
- $\frac{\Delta}{2} < k \leq \Delta$: the infection spreads at the maximum possible rate (flooding) and it is impossible to leave uninfected gaps.
- $2 \leq k \leq \frac{\Delta}{2}$:
  - permanently uninfected nodes must be permanently unreachable from all active nodes
  - try to construct two blocks of inactive (previously infected) nodes with isolated uninfected nodes between the blocks
  - blocks must contain at least $\frac{\Delta}{2}$ nodes to prevent an active node from infecting an isolated node
  - two possible strategies (both fail)

Protection near Originator



Attempt to isolate uninfected nodes close to originator

Protection far from Originator



Attempts to isolate uninfected nodes far from originator