On the fixed points of the iterated pseudopalindromic closure

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Outline

1. Iterated palindromic closure and Sturmian words
2. Generalization to an alphabet with more than 2 letters
3. Generalization using pseudopalindromes
4. Existence of fixed points of the iterated pseudopalindromic closure
5. Properties of the fixed points
6. Open problems
Iterated palindromic closure

Palindromic closure

The *palindromic closure* of $w \in \mathcal{A}^*$, denoted $w^+$, is the shortest palindrome having $w$ as prefix.

Iterated palindromic closure

Let $w \in \mathcal{A}^*$. The iterated palindromic closure of $w$, denoted $\text{Pal}(w)$, is defined by

$$\text{Pal}(\varepsilon) = \varepsilon, \quad \text{Pal}(w) = (\text{Pal}(w[1, n-1]) \cdot w[n])^+.$$

Example

$$\text{Pal}(123) = (\text{Pal}(12)3)^+ = ((\text{Pal}(1)2)^+3)^+ = ((12)^+3)^+ = ((121)3)^+ = 1213121.$$
Generalization of the iterated palindromic closure

It can naturally be generalized to an infinite word $u \in \mathcal{A}^\omega$ as

$$\text{IPal}(u) = \lim_{n \to \infty} \text{Pal}(u[1, n]).$$

Standard Sturmian and standard episturmian words

Let $w \in \mathcal{A}^\omega$. Then $\text{IPal}(w)$ is a \textit{standard episturmian word} if $|\mathcal{A}| \geq 3$ and if $|\mathcal{A}| = 2$ and $w \neq u\alpha^\omega$, it is a \textit{standard Sturmian word}.

Example

Let consider the Tribonacci word $T = \text{IPal}((123)^\omega)$. Then,

$$T = 121312112131212131211213121 \cdots$$

We denote by $\Delta(T)$ the word that determines $T$, called the \textit{directive word} of $T$. Here, $\Delta(T) = (123)^\omega$. 
Pseudopalindrome

An antimorphism of $\mathcal{A}^*$ is a function $\theta : \mathcal{A}^* \rightarrow \mathcal{A}^*$ such that for all $u, v \in \mathcal{A}^*$, $\theta(uv) = \theta(v)\theta(u)$.

If $\theta^2 = \text{id}$, then it is involutive.

### $\theta$-palindrome

For a fixed involutive antimorphism $\theta$, a finite word $w \in \mathcal{A}^*$ is called a $\theta$-palindrome (pseudopalindrome) if $\theta(w) = w$.

In the sequel, $R : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is the involutive antimorphism defined as the reversal.

#### Example

For $w = abcab$, $R(w) = bacba$.

The $R$-palindromes are exactly the usual palindromes.
Involution antimorphisms

**Lemma**

Let $\tau$ be an involutive permutation over the alphabet $A$. Then $\theta = \tau \circ R = R \circ \tau$ is the unique involutive antimorphism on $A^*$ that extends $\tau$. Thus,

$$\theta(w) = \tau(w[n])\tau(w[n-1]) \cdots \tau(w[1]).$$

Any involutive antimorphism can be obtained that way.

**Example**

Let $\theta = R \circ \tau$, with $\tau(a) = b$ and $\tau(b) = a$. Then $abab$ is a $\theta$-palindrome. Indeed, $abab$ is a fixed point under $\theta$:

$$\theta(abab) = R(baba) = abab.$$
Iterated pseudopalindromic closure

The \( \theta \)-palindromic (pseudopalindromic) closure of a word \( w \in \mathcal{A}^+ \), denoted \( w^\oplus \), is the shortest \( \theta \)-palindrome having \( w \) as prefix.

**Example**

Let \( \theta \) be an involutive antimorphism such that \( \tau(1) = 3 \), \( \tau(2) = 4 \), \( \tau(3) = 1 \) and \( \tau(4) = 2 \), and let \( w = 123124 \). Then

\[
w^\oplus = 1231 \cdot 24 \cdot \theta(1231) = 1231 \cdot 24 \cdot 3143.
\]

The *iterated pseudopalindromic closure*, denoted \( \text{Pal}_\theta \), is naturally defined by \( \text{Pal}_\theta(\varepsilon) = \varepsilon \), and for \( w \in \mathcal{A}^* \),

\[
\text{Pal}_\theta(w) = (\text{Pal}_\theta(w[1, n-1])w[n])^\oplus.
\]

And \( \text{IPal}_\theta = \lim_{n \to \infty} \text{Pal}_\theta(w[1, n]) \).
Questions

1. What do the fixed points of the iterated pseudopalindromic closure look like?
2. How many are they?
3. Do they have remarkable combinatorial properties?
Existence (1/2)

Example

\[
\text{IPal}_R(abx\cdots) = abax\cdots \\
\text{IPal}_R^2(abx\cdots) = ababaax\cdots \\
\text{IPal}_R^3(abx\cdots) = abababaababaababax\cdots.
\]

Let \( E \) be the involutive antimorphism defined by \( E = R \circ \tau \), with \( \tau(a) = b \) and \( \tau(b) = a \). Then

\[
\text{IPal}_E(abx\cdots) = abbaabx\cdots \\
\text{IPal}_E^2(abx\cdots) = abbaabbababbaabbaabbaabbaabbaabbaabbaaabcd\cdots.
\]
Existence (2/2)

Theorem and definition

Over a $k$-letter alphabet, with $k \geq 2$, there are 3 kinds of fixed points having at least 2 different letters, only depending on the first letters of the word and the involutory antimorphism $\theta = R \circ \tau$ considered.

1. When $\tau(a) = a$ and $\tau(b) = b$, with $a \neq b$, for a fixed $n \geq 1$, $\text{IPal}_\theta$ has a unique fixed point beginning with $a^n b$, denoted $s_{R,n,a,b}$, which equals

$$s_{R,n,a,b} = \lim_{i \to \infty} \text{Pal}_i^i(a^n b) = a^n b a^n (aba^n)^{n+1} b(a^{n+1}b)^{n+1} a^n a \cdots .$$

2. When $\tau(a) = a$ and $\tau(b) = c$ for pairwise different letters $a, b, c$, for a fixed $n \geq 1$, $\text{IPal}_\theta$ has a unique fixed point beginning with $a^n b$, denoted by $s_{H,n,a,b,c}$, which equals

$$s_{H,n,a,b,c} = \lim_{i \to \infty} \text{Pal}_H^i(a^n b) = a^n b c a^n c b a^n b c a^n (ab c a^n c b a^n b c a^n)^{n+1} c \cdots .$$

3. When $\tau(a) = b$ and $\tau(b) = a$, with $a \neq b$, $\text{IPal}_\theta$ has a fixed point beginning with $a^n b$ only if $n = 1$. It is denoted by $s_{E,a,b}$ and equals

$$s_{E,a,b} = \lim_{i \to \infty} \text{Pal}_E^i(a) = abbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaabbaab.
Combinatorial properties of $s_{R,n,a,b}$

1. For a fixed positive $n \in \mathbb{N}$, $s_{R,n,a,b}$ is not ultimately periodic and consequently, is standard Sturmian (using Droubay, Justin, Pirillo - 2001).

2. For a fixed $n \in \mathbb{N}$, $s_{R,n,a,b}$ is not a fixed point of a nontrivial morphism (using Arnoux, Rauzy - 1991 and Crisp et al - 1993).

3. $s_{R,n,a,b}$ is $(n + 4)$-th power-free, but contains $(n + 3)$-th powers (using Vandeth - 2000).

4. For any $n \geq 1$, $\alpha_{n,a,b}$ is transcendental (using Adamczewski, Bugeaud - 2007).
Combinatorial properties of $s_{E,a,b}$ (1/2)

**Useful property [de Luca, De Luca - 2006]**

Let $\theta = \tau \circ R$ be an involutory antimorphism over an alphabet $A$, with $\mu_\theta$ the morphism defined for all $a$ in $A$, by $\mu(a) = a$ if $a = \tau(a)$ and by $\mu(a) = a\tau(a)$ otherwise.

Then, for any $w \in A^\omega$ and for any involutory antimorphism $\theta$, one has

$$\text{IPal}_\theta(w) = \mu_\theta(\text{IPal}(w)).$$

**Idea**

First consider $w_E = \text{IPal}(s_{E,a,b})$ and then, extend its properties to

$$\mu_E(\text{IPal}(s_{E,a,b})) = s_{E,a,b}.$$

$$w_E = ababaababaababaababaababaababaababaababaababaababa\ldots$$
Combinatorial properties of \( s_{E,a,b} \) (2/2)

**Property of \( w_E \)**

\( w_E \) is not ultimately periodic, and consequently, is a Sturmian word.

**Lemma**

An infinite word \( w \) over \( A \) is ultimately periodic if and only if \( \mu_\theta(w) \) is so.

\[ \implies s_{E,a,b} \text{ is not ultimately periodic.} \]

**Lemma**

\( w_E \) is not a fixed point for some non-trivial morphism \( \implies s_{E,a,b} \) is not a fixed point for some non-trivial morphism.

**Proposition**

\( w_E \) and \( s_{E,a,b} \) both contain 4-th powers, but no 5-th power words (using Shur - 2000).
Combinatorial properties of $s_{H,n,a,b,c}$ (1/3)

Recall: $\tau(a) = a$, $\tau(b) = c$ and $\tau(c) = b$, and $\theta = R \circ \tau$.

First consider $w_H = \text{IPal}(s_{H,n,a,b,c})$ and then extend its properties to $\mu_H(\text{IPal}(s_{H,n,a,b,c})) = s_{H,n,a,b,c}$.

$$w_{H,n} = \text{IPal}(s_{H,n,a,b,c}) = a^n ba^n ca^n ba^n aba^n ca^n ba^n \cdots$$

**Property of $w_{H,n}$**

$w_{H,n}$ is not ultimately periodic, and consequently, is a strict standard episturmian word.

$\implies s_{H,n,a,b,c}$ is not ultimately periodic.

**Proposition**

$w_{H,n}$ is not a fixed point of a nontrivial morphism (using Justin, Pirillo - 2002).
Combinatorial properties of $s_{\mathcal{H},n,a,b,c}$ (2/3)

Generalization of a result of Justin, Pirillo - 2002 for strict episturmian word having periodic directive word:

**Proposition**

Let $s$ be a strict standard episturmian word directed by a word $\Delta$ and let $\ell$ denotes the greatest integer such that $\alpha^{\ell}$ is a factor of $\Delta$ with $\alpha$ a letter. Assume $\Delta$ contains at least one factor $aua^{\ell}va$ with $a$ a letter and $u,v$ non empty words that do not contain the letter $a$. Then $s$ is $(\ell + 3)$-th power-free but contains an $(\ell + 2)$-th power.

**Corollary**

The words $w_{\mathcal{H},n}$ are $(n + 4)$-th power free but contain $(n + 3)$-th powers.
Combinatorial properties of $s_{H,n,a,b,c} (3/3)$

Let $s_{H,n,a,b,c}$ be a fixed point of the $\mathrm{IPal}_H$ operator, for a fixed $n$. Then $s_{H,n,a,b,c}$ satisfies the following properties:

1. It is not an episturmian word, but is a pseudostandard word.
2. It is not a fixed point for some non trivial morphism.
3. It is $(n + 4)$-th power-free but contains $(n + 3)$-th powers.
4. The frequencies of the letters $b$ and $c$ are equal.
Open problems

- What is the critical exponent: its value, the number of occurrences, their positions, etc.
- Does there exist a connection between $s_{R,n,a,b}$ and $s_{R,n+1,a,b}$? between $s_{H,n,a,b,c}$ and $s_{H,n+1,a,b,c}$?
- Can we give a geometric interpretation of the iterated palindromic (pseudopalindromic) closure?
- What are the letter frequencies of $s_{H,n,a,b,c}$?
- Any remarkable combinatorial properties?