2nd CanaDAM Conference Open Problems

Edited by Brett Stevens*

Abstract

Open problems submitted to the 2nd Canadian Discrete and Algorithmic Mathematics Conference will be held on May 25-28, 2009, at the Centre de recherches mathématiques in Montréal, Canada.

put brief summary here

Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Please send other comments and information about partial or full solutions to the organization committee for the next CanaDAM Conference.

PROBLEM 1. Chemical Graph Theory: Maximum cardinality resonant sets and maximal alternating sets of hexagonal systems

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Let H be a hexagon system: that is, a 2-connected plane graph whose inner faces are all regular hexagons.

Let P be a subset of the hexagons of H. If we can find a perfect matching M such that each of the hexagons of P intersects M in precisely three edges, then we say that P is a *maximal alternating set*.

An *M-resonant set* S is a set of pairwise disjoint hexagons in H such that the boundary of each of the hexagons in S is alternating in M.

Conjecture 1: Let H, P, M be given as defined. Then an M-resonant set of maximum cardinality is canonical, in the sense that if you delete the vertices of the M-resonant set from H, the remaining graph will either be empty or it will have a unique perfect matching.

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PROBLEM 2. (d,k)-radially Moore Graphs with more than one central vertex

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Let G be a digraph. First some definitions:

- The eccentricity of a vertex u in a digraph G is the maximum between:
  - The maximum distance from u to any other vertex in G,
  - The maximum distance to vertex u from any other vertex in G.
- The radius of G, is the *minimum* eccentricity among all vertices of G.
- The diameter of G is the *maximum* distance between any pair of vertices in G.
- The center of G is the set of all vertices with minimum eccentricity.

Problem 1: A regular digraph of degree $d > 1$, radius $k > 1$, diameter $k+1$ and order $n_{d,k} = 1 + d + d^2 + \cdots + d^k$ (Moore bound) is called a (d,k)-*radially Moore digraph*. It is known that radially Moore digraphs exist for any value of the degree d and the radius k. Nevertheless, any radially Moore digraph known until now contains at most one central vertex when $d > 1$ and $k > 2$. Can you find a (d,k)-radially Moore with more than one central vertex for $d > 1$ and $k > 2$?

PROBLEM 3. Spherical Codes

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Let G be a finite group, $g = |G|$, $\rho : G \to GL(\mathbb{R}^d)$ is a representation of G, $v \in \mathbb{R}^d$, and S is the orbit of v under $\rho(G)$. We call $S = G \cdot v$ the *group code* for G in $\mathbb{R}^d$. Suppose also that $\pi : \mathbb{R}^d \to \mathbb{R}^d$ is a rank-k linear idempotent map, i.e. π is the orthogonal projection onto a k-dimensional subspace.

Problem 1: Determine a lower bound on the cardinality $|\pi(S)|$ in terms of g, $|S|$, d, and k.

Example: If S is the set of vertices of a 3-dimensional cube, then every 2-dimensional projection of S has at least 4 points.

PROBLEM 4. Generators in arithmetic progression

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Problem 1: Let p be an odd prime. Can there exist four elements of $\mathbb{U}(p)$ (the group of units of $\mathbb{Z}_p$), say, w, x, y, z, such that

- the cyclic subgroups generated by w, x, y, z are non-trivial;
- $\mathbb{U}(p) = \langle w \rangle \times \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ (direct product);
- w, x, y, z form an arithmetic progression?

No solutions are currently known; Peter J. Cameron and Donald Preece have checked all primes up to 10000.

Example: Many examples are known for three factors instead of four, the smallest is

$$\mathbb{U}(31) = \langle 25 \rangle \times \langle 30 \rangle \times \langle 4 \rangle,$$

where the factors have orders 3, 2, 5 respectively. Solutions with four factors are known with trivial factors, the smallest is

$$\mathbb{U}(3613) = \langle 3528 \rangle \times \langle 1148 \rangle \times \langle 2381 \rangle \times \langle 1 \rangle,$$

where the factors have orders 4, 129, 7 and 1 respectively.

PROBLEM 5. $1+1/k$ approximation algorithm for weighted k-edge connected spanning multisubgraph problem

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Let OPT be the optimal value of the following linear program, which has a variable x_e for each edge of a graph $G = (V, E)$.

$$\begin{array}{rcl} \text{minimize} & c \cdot x & \text{subject to} \\ x & \geq & 0 \\ \displaystyle\sum_{\substack{S \subset V \\ S \neq \emptyset, V \\ e \in \Delta(S)}} x_e & \geq & 1 \end{array}$$

Let $min - kECSS$ be the minimum cost of a k-edge connected spanning multisubgraph of G.

Problem 1: Is $min - kECSS \leq (k+1)OPT$ for all k?

- This is known to be true for $k = 1$
- Showing $min - kECSS \geq k\dot{O}PT$ is easy, which is why we might hope to get an approximation algorithm.

PROBLEM 6. Partitions into matchings and edge covers

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Let $M = (V, E)$ denote a multigraph, d be the minimum degree and D be the maximum degree.

- E' is a *matching* if $D(v, E') \leq 1$
- E' is an *edge cover* if $d(V, E') \geq 1$.

Theorem 1 (Shannon (1949))**.** *The minimum $f(D)$ so that for all M, $E(M)$ can be partitioned into $f(D(M))$ matchings, is $\lfloor 3D/2 \rfloor$.*

Problem 1: Find the maximum $g(d)$ so that for all M, $E(M)$ can be partitioned into $g(d(M))$ edge covers.

Comment: I can show

- $g(d) \leq \lfloor 3d/4 + 1/4 \rfloor, \forall d$

- $g(d) \geq \lfloor 3\lfloor d/2 \rfloor /2 \rfloor, \forall d.$

which match except on odd d, where they differ by 1.

PROBLEM 7.

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Let $I(n,r)$ be the number of permutations of n objects with r inversions. Then $I(n,r)$ is the coefficient of x^r in $(1+x)(1+x+x2)...(1+x2+...+xn-1)$.

Problem 1: Find a *simple* non-recursive formula for $I(n,r)$.

Comment: Actually, I did find such a formula in 1966, but it is so complicated that my article presenting it was condemned, derided and dismissed by both referees. A special case of this formula, where $r \leq n$, appeared in 1973 in [1]. I finally managed to publish my monstrosity of a formula by sneaking it into an article about something entirely different [2]. So the problem I'm actually presenting is to find a formula for $I(n,r)$ that is simple enough to be useful for something besides making referees feel superior to authors.

***References**

[1] D. E. Knuth, The Art of Computer Programming Volume 3. Addison-Wesley Reading Mass., 1973 p. 16.

[2] T.R. Walsh, Loop-free sequencing of bounded integer compositions, *The Journal of Combinatorial Mathematics and Combinatorial Computing.*, 33 (2000) 323–345.

PROBLEM 8. A de Bruijn - Erdős theorem in metric spaces?

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De Bruijn and Erdős [3] proved that every noncollinear set of n points in the plane determines at least n distinct lines. Chen and Chvátal [1] suggested that this theorem might generalize:

Problem 1: True or false? Every finite metric space (X,d) where no line consists of the entire ground set X determines at least $|X|$ distinct lines.

Here, the *line* $\overline{uv}$ in a metric space is defined as the union of $\{u, v\}$ and all the pins containing $\{u, v\}$, where a *pin* is any three-point set $\{x, y, z\}$ such that $d(x, y)+d(y, z) = d(x, z)$.

It is known that

- in every metric space on n points, there are at least $\lg n$ distinct lines or else some line consists of all n points [1];
- in every metric space on n points, there are $\Omega((n/\rho)^{2/3})$ distinct lines, where ρ is the ratio between the largest distance and the smallest nonzero distance [2];
- in every metric space induced by a connected graph on n vertices, there are $\Omega(n^{2/7})$ distinct lines or else some line consists of all n vertices [2];
- in every metric space on n points where each nonzero distance equals 1 or 2, there are $\Omega(n^{4/3})$ distinct lines and this bound is tight [2].

***References**

[1] X. Chen and V. Chvátal, "Problems related to a de Bruijn - Erdős theorem", *Discrete Applied Mathematics* 156 (2008), 2101 – 2108.

[2] E. Chiniforooshan and V. Chvátal, "A de Bruijn - Erdős theorem and metric spaces", arXiv:0906.0123v1 [math.CO]

[3] N.G. de Bruijn and P. Erdős, On a combinatorial problem, *Indagationes Mathematicae* 10 (1948) 421–423.

PROBLEM 9. Problem on unit distance graphs in n-dimensional spaces

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Fix a rational point $x = (a/n, b/n) \in \mathbb{R}^2$, where the denominator n is a product of primes congruent to 1 (mod 4). When $|x| < 1$, we can always find a path, P, composed of unit line segments from x to the origin, such that the endpoints of the each line segment of the path are both rational.

Problem: Is there some constant bound on the length of this path (as a function of n)?

Comment: If our points are in $\mathbb{R}^5$, the answer is yes, but we really want to know the answer for $\mathbb{R}^2$.

PROBLEM 10. On the Hadwiger conjecture

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We say that a graph H has the *Hadwiger property* if the following holds:

For every graph G, if there is a graph homomorphism $G \to H$ but for any subgraph $H' \subset H$, $G \not\to H'$, then G contains H as a minor.

Conjecture (The Hadwiger conjecture): Every complete graph has Hadwiger property.

Comment: The complete graphs K_i, $i \leq 6$, do have the Hadwiger property. It is also very easy to check that every odd-cycle has Hadwiger property.

In general if a graph H has the Hadwiger property then it must be a core. But not every core has Hadwiger property. For example W_5 does not have Hadwiger property. If G is a graph obtained from Hajos sum of two copies of K_4 then G maps to W_5 but to no proper subgraph of W_5 (G and H are both are 4-critical).

Problem: Is there any other graph that has Hadwiger Property? If so what are the sufficient conditions for a graph to have Hadwiger Property?

PROBLEM 11. Some Old Problems Worth Reviving

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These are old problems that I feel are worth reviving. All graphs are simple, undirected, finite, unless otherwise stated. And edge between u and v is denoted by $[u, v]$ and t and the cardinality of the vertex set by n.
The currency used is *Erdős dollars*, that is, US dollars that he offered to pay for solutions.

Problem: (Erdős, Farber, Lovász, about 1975, 500E$) Let $n \in \mathbb{N}$, $n > 0$. For $1 \leq i \leq n$ let K_n^i be a complete graph on n vertices such that $|V(K_n^i) \cap V(K_n^j)| = 1$ (any two of the graphs intersect in exactly one vertex). Let $G = \cup_{i=1}^n K_n^i$. Prove that the chromatic number of G is n.
Note. Jeff Kahn proved a fractional version of this result. The original problem is devilishly hard and open.

Problem: Let $G\square H$ be the box (Cartesian) product of two graphs G and H (i.e. $V(G\square H) = V(G) \times V(H)$ and $E(G\square H) = \{[(u,x),(v,y)] : \textit{either } u = v, [x,y] \in E(H), \textit{ or } [u,v] \in E(G), x = y\}$). When $G = H$, define G^k ($k \geq 2$) by $G^k = G\square G^{k-1} = G^{k-1}\square G$, with $G^1 = G$. Give good bounds on the stability number $\alpha(G^k)$ in terms of $\alpha(G), |V(G)| = n, k$.
Note. The lower bound should be better than $(\alpha(G))^2$ and the upper bound better than $\alpha(G)n^{k-1}$.

Problem: For any graph G, let $I(G) = \lim_{k\longrightarrow\infty} \frac{\alpha(G^k)}{n^k}$. This limit always exists. Find $I(W_5)$.
Note. The wheel W_5 consists of a cycle of length 5 and an additional vertex adjacent to the five vertices of the cycle. See
G.Hahn, P.Hell, S.Poljak, On the ultimate independence ratio, *European Journal of Combinatorics* **16** (1995), 253 – 261.

Problem: A graph is *cop-win* if in the classic game of cop-and-robber (see note) the cop has a winning strategy. Find examples of infinite cop-win graphs that are not trivially based, that is, such that trivially cop-win graphs are not used in their construction. By "trivially cop-win" we mean trees of bounded diameter and graphs containing a universal vertex.
Note.

***References**

[1] **R. Nowakowski and P. Winkler, "Vertex-to-vertex pursuit in a graph", *Discrete Math.* 43 (1983), 235–239.**

[2] **G. Hahn, "Cops, robbers and graphs", *Tatra Mountains Mathematical Publications* 36 (2007), 1 – 14**

[3] **G. Hahn, F. Laviolette, N. Sauer, R.E. Woodrow, "On cop-win graphs", *Discrete Math.* 258 (2002), 27 – 41**

[4] **A. Bonato, G. Hahn, C. Tardif, "Large classes of infinite k-cop-win graphs", submitted.**