Computing Fault Tolerance of Cayley Graphs

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Outline

- Cayley Graphs and Computational Problems
- Fragments and Atoms
- Exchange Graphs
- Network Flow and Algorithm

To define a Cayley graph we need a group G and a subset $S \subseteq G$.

G: any group

S : any subset of G not containing the identity.

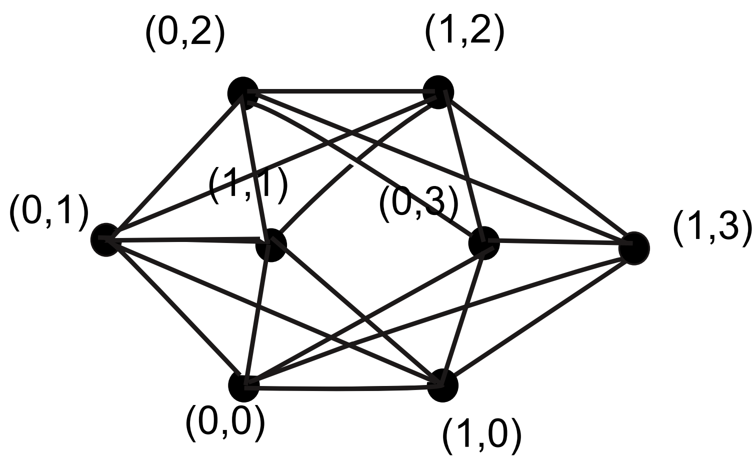
Cayley graph (G, S): elements of G are vertices and, for $x, y \in G$, there is a directed edge from x to y iff $x \cdot s = y$ for some $s \in S$.

Examples: cycles (directed and undirected), Hypercubes, truncated hypercubes, etc.

Examples of Cayley graphs (G, S).

Consider $(\mathbb{Z}_5^*, S)$ with $S = \{2, 3\}$.

Consider $(\mathbb{Z}_2 \times \mathbb{Z}_4)$ with $S = \{(1,0), (0,1), (0,3), (1,3), (1,1)\}$.



Examples

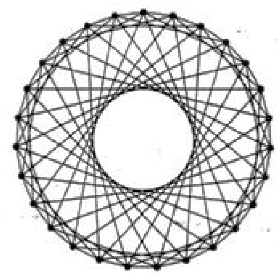


Figure 1: $G = \mathbb{Z}_{27}$, $A = \{1, 4, 17\}$ and $S = A \cup A^{-1}$. (D.F.Hsu)

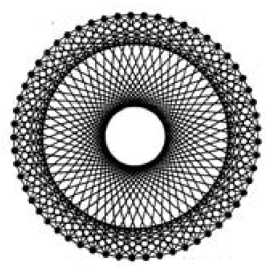


Figure 2: $G = \mathbb{Z}_{57}$, $A = \{1, 13, 33\}$ and $S = A \cup A^{-1}$. (D.F.Hsu)

Properties of Cayley graphs make them good candidates for communication networks.

- Regular: each vertex has out-degree and in-degree $|S|$.
- Vertex transitive, provided it is strongly connected, that is, every element of G can be written as a product of elements from S.
- They have small degree and small diameters.
- They are useful for constructing good expander graphs.

Fault tolerance and vertex connectivity are essentially the same.

The **fault tolerance** of a digraph $\mathcal{X}$ is the largest number k such that failure of k nodes does *not* destroy the connectivity of the whole network.

If $\kappa(\mathcal{X})$ is the cardinality of the smallest vertex cut then the fault tolerance of $\mathcal{X} = \kappa(\mathcal{X}) - 1$.

(G, S) has **optimal fault tolerance** when the smallest vertex cut has cardinality $|S|$.

Computational Problems

Problem 1. Given a finite group G and a subset S of G, decide if the Cayley graph (G, S) is strongly connected.

Problem 2. Given a finite group G and a subset S of G, compute the fault tolerance of (G, S), assuming the graph is strongly connected.

"Given a finite group G": We assume that the group G is given by an *oracle* (or a *black box*). The oracle can perform various group operations, namely, product of two elements, the inverse of an element, and distinctness of two elements.

Question: Are there polynomial time algorithms for the above problems?

We need to be careful about what we consider polynomial time.

Polynomial time: the number of group operations used is bounded by a polynomial in $|S|$ and $\log |G|$.

Warning: One can not examine all the vertices in the graph!

Answers

Problem 1: **still open**

It is open even for the special case: G is the multiplicative group of a finite field $\mathbb{F}_q$ and $S = \alpha$. In this case, the graph (G, S) is strongly connected iff α a primitive element (i.e. α has multiplicative order $q - 1$):

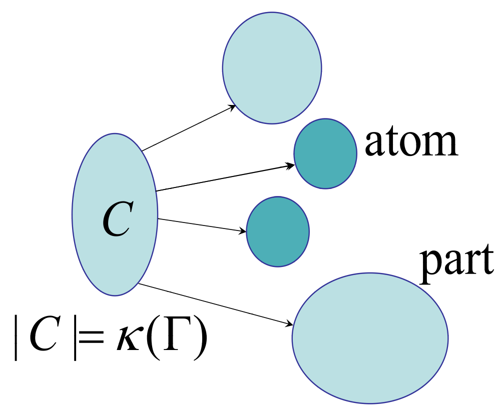
> Consider $G = \mathbb{F}_{2^n}^*$, $n = 10,000$. $\mathbb{F}_{2^n} = \mathbb{F}_2[x]/f(x)$ where $f(x)$ is irreducible of degree n.
>
> α is presented in the basis $(1, x, \ldots, x^{n-1})$ mod $f(x)$.
>
> $(G, \{\alpha, \alpha^{-1}\})$ is connected iff has order $2^n - 1$.

Problem 2: **yes**

To see how to answer problem 2 we need to ‘fragments’ and ‘atoms’.

Watkins (1970): For any digraph $\mathcal{X}$ and any vertex cut $C \subseteq V(\mathcal{X})$, the strongly connected components of $\mathcal{X} \setminus C$ are called the **fragments** of $\mathcal{X}$ induced by C.

A fragment is called an **atom** if it is induced by a minimum vertex cut and it has minimum cardinality among all such fragments.



The are two ways a vertex set can be a fragment.

For any subset A of $V(\mathcal{X})$, we denote

$$N^+(A) = \{v \in V(\mathcal{X}) \setminus A : [u,v] \in E(\mathcal{X}) \text{ for some } u \in A\},$$
$$N^-(A) = \{v \in V(\mathcal{X}) \setminus A : [v,u] \in E(\mathcal{X}) \text{ for some } u \in A\},$$

called **the positive or negative neighborhood** of A, respectively.

If A is an atom, then $N^+(A)$ or $N^-(A)$ is a vertex cut (of minimum cardinality), called *positive* or *negative* atom, respectively.

Three Structural Theorems

Theorem 1 (W. Watkins 1970 and Y.O. Hamidoune 1977): Let $\mathcal{X}$ be any strongly connected vertex transitive digraph with a positive atom. Then its positive atoms form a partition of all the vertices.

Theorem 2 (Y.O. Hamidoune 1984): Assume that the Cayley graph (G, S) is strongly connected and contains positive atoms. Let A be the positive atom of (G, S) containing 1. Then $A = \left\langle S \cap A \right\rangle$ and every positive atom is of the form aA, $a \in G$, i.e. a left coset of A.

Theorem 3 (Gao and N. 2007): $A \subseteq S \cdot S^{-1} = \{a \cdot b^{-1} : a, b \in S\}$.

Consequences of our structural theorem

- A very simple proof that ‘exchange graphs’ are optimally fault tolerant.

- An efficient algorithm for computing fault tolerance in connected Cayley graphs. (Polynomial in an appropriate sense.)

Exchange Graphs

C. Godsil (1981):
S_n: the symmetric group of permutations on $\{1, 2, \cdots, n\}$.
Γ: any graph (undirected) on the vertex set $\{1, 2, \cdots, n\}$.

Each edge (i, j) of Γ corresponds to a transposition in S_n that exchanges i and j.

Fact. (S_n, Γ) is connected iff Γ is connected.

Exchange Graphs have Optimal Fault Tolerance

Theorem 4 (Gao, N.). The connectivity of (S_n, Γ) is equal to $|E(\Gamma)|$, the number of edges in Γ.

Proof. Suppose $\kappa(S_n, \Gamma) < |E(\Gamma)|$. Then the atom A containing 1 has size at least 2 and is a subset of

$$\Gamma \cdot \Gamma^{-1} = \{(i,j)(a,b) : (i,j),(a,b) \in E(\Gamma)\}.$$

Furthermore, A is generated by $A \cap \Gamma$ and, as $|A| \geq 2$, in particular $A \cap \Gamma \neq \emptyset$. Thus there is a 2-cycle of Γ that lies in $\Gamma \cdot \Gamma^{-1}$, impossible.

Network Flow and Algorithm

We assume the the Cayley graph is connected, or equivalently work with the connected component containing 1.

Let G be any finite group and $S \subset G$ not containing 1.

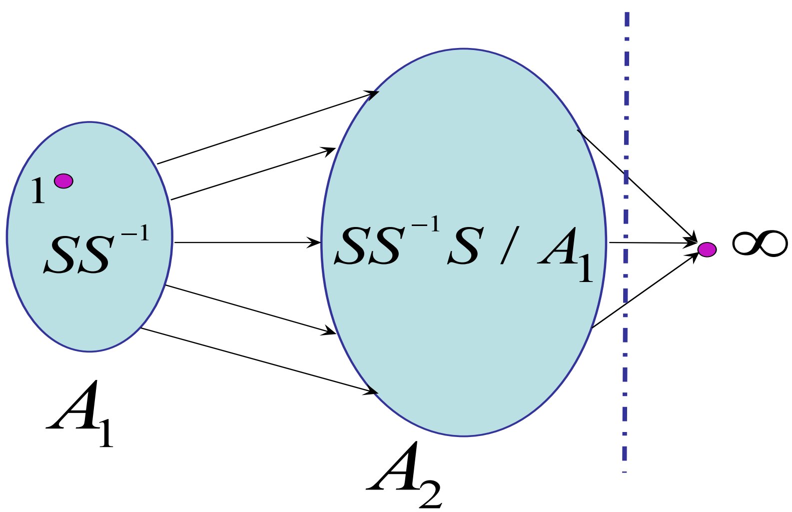
Let G_0 be the subgroup generated by S. Then the Cayley graph (G_0, S) is the connected component of $\mathcal{X}$ that contains the identity 1. **We denote this component by $\mathcal{X}_0$, i.e., $\mathcal{X}_0 = (G_0, S)$.**

We create the smaller graph, $\bar{X}_0$.

Lemma. Suppose $G_0 \neq A_1 \cup A_2$. Then $\kappa(\mathcal{X}_0)$ is equal to the maximum flow from 1 to v_∞ in $\overline{\mathcal{X}}_0$ (with each edge of capacity 1).

Proof. We show $\kappa(X_0) = \kappa(\bar{X}_0)$:

Create the smaller graph, $\bar{X}_0$.



$A \subseteq SS^{-1}$ (Using structural theorem.)
$N^+(A) \subseteq A_1 \cup A_2$

Find maximum flow in $\bar{X}_0$.

Network Flow and Algorithm

Algorithm:

Input: a black box (oracle) for a group G and $S \subset G$

Output: Fault tolerance of the connected components of the Cayley graph (G, S)

Step 1: Compute the vertices in A_1 and A_2

Step 2: If $G = A_1 \cup A_2$ then compute the connectivity of (G, S), say k.

Step 3: Otherwise construct the network $\overline{\mathcal{X}}_0$.

Step 4: Find the maximum flow from 1 to v_∞, say k.

Return $k-1$.

Note that the network $\overline{\mathcal{X}}_0$ has at most $|S|^3+1$ vertices. So the algorithm runs in polynomial time.

Hence we have solved our Problem 2.

Theorem 5(Gao, N.) If a strongly connected Cayley digraph $X = (G, S)$ is given by the set S together with a black box that efficiently provides inverses, multiplication and recognition of the identity element, then the connectivity $\kappa(X)$ may be determined in time polynomial in $|S|$ and $\log|G|$.

In other words the number of calls to the oracle is at most $|S|^c$ for some constant c.

Open Problems

Fact. Computing fault tolerance of Cayley graphs is easy!

Open Problem 1: Is there a polynomial time algorithm to decide the connectedness of Cayley graphs!

Open Problem 2: Which Cayley graphs are Hamiltonian?

Open Problems

Open Problem 3: Star diameters and routing on Cayley graphs?

Shuhong Gao, Beth Novick and Ke Qiu, *From Hall's Matching Theorem to Optimal Routing on Hypercubes*, Journal of Combinatorial Theory, Series B 74, 291-301 (1998).

Shuhong Gao and D. Frank Hsu, *Short containers in Cayley graphs*, to appear in Discrete Applied Mathematics.