TP_2 Completion Problem

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Outline

- TP_2 Completion
  - ☞ TP_2 Completion Problem
- TP_2 Partial Order and Bruhat Partial Order
  - ☞ TP_2=Bruhat

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- *TP_2 Matrix:* An $m \times n$ matrix in which every minor of size at most 2 is positive.
- *Partial TP_2 Matrix:* An $m \times n$ matrix in which some entries are specified and the remaining entries are unspesified, and every minor of specified entries of size at most 2 is positive.

$$\begin{pmatrix} 2 & ? & 1 \\ ? & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

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- *TP_2 Comlpletion:* Replacing unspecified entries by numbers such that the resulting matrix is TP_2.
- $$\begin{pmatrix} 2 & ? & 1 \\ ? & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

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- $$\begin{pmatrix} 2 & ? & 1 \\ ? & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

- *TP_2 Comlpletion Problem:* For which patterns, does every partial TP_2 matrix have a TP_2 completion?
- *Example:*

$$\begin{pmatrix} \times & ? & \times \\ ? & \times & \times \\ \times & \times & \times \end{pmatrix}$$

- $$\begin{pmatrix} 1 & x & 1 \\ y & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \qquad x = y = 1, \text{ TP}_2 \text{ Completable}$$

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$$\mathcal{P} = \begin{pmatrix} a_{11} & x_{12} & a_{13} \\ x_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Partial TP_2 matrix $\mathcal{P}$ is TP_2 completable iff $a_{11}a_{23}a_{32} > a_{13}a_{22}a_{31}$.

Bruhat Order on Permutations

- *Upward Transposition:* for $\pi \in S_n$, a transposition of i and j when $i < j$ and $\pi(i) < \pi(j)$.
- *Bruhat Order:* Suppose $\pi \neq \sigma \in S_n$. If σ can be obtained from π by a sequence of upward transpositions, then π is said to be less than σ in the *Bruhat partial order*. We denote this by $\pi <_{Br} \sigma$.
- *Example:*

$$2143 <_{Br} 2413 <_{Br} 3412$$

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Notations

- Let π be a permutation on $I \subseteq \{1, 2, \ldots, n\}$, then $M_\pi = [m_{ij}] \in M_n$ defined as following

$$m_{ij} = \begin{cases} 1, & \text{if } j = \pi(i) \text{ for } i \in I \\ 0, & \text{otherwise} \end{cases}$$

Notations

- $M_\pi[p,q]$: the number of ones in the submatrix of M_π lying in the rows $1,2,\ldots,p$ and columns $1,2,\ldots,q$.

$$M_\pi = \begin{array}{c} \\ 1 \\ \vdots \\ \\ p \\ \\ \end{array} \overset{\begin{array}{ccc} 1 & \ldots & q \end{array}}{\left(\begin{array}{ccccc} * & \ldots & * & & \\ * & \ldots & * & & \\ & \vdots & & & \\ * & \ldots & * & & \\ & & & & \end{array} \right)}$$

Lemma (known)

For two permutations $\sigma \neq \pi \in S_n$, $\pi <_{Br} \sigma$ if and only if $M_\pi[i,j] \geq M_\sigma[i,j]$ for all $i,j \in \{1,2,\ldots,n\}$.

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- Given a matrix $A \in M_n$ and a permutation $\pi \in S_n$, let

$$A_\pi = \prod_{i=1}^{n} a_{i\pi(i)}.$$

- *TP_2 Partial Order:* If $\pi, \sigma \in S_n$ are such that

$$A_\pi < A_\sigma, \; \forall A \in \mathrm{TP}_2(n,n),$$

then π is less than σ in the *TP_2 partial order*. We denote this as

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Theorem

For $\pi, \sigma \in S_n$, $\pi \neq \sigma$, we have $\sigma <_{Br} \pi$ if and only if $\pi <_{TP_2} \sigma$.

- $\pi <_{TP_2} \sigma \Rightarrow \sigma <_{Br} \pi$

$$K = \left[\begin{array}{c|c} 2J & J \\ \hline J & J \end{array}\right]$$

$$2^{M_\sigma[p,q]} = \prod_{l=1}^{n} k_{l\sigma(l)} < \prod_{l=1}^{n} k_{l\pi(l)} = 2^{M_\pi[p,q]}$$

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Theorem

A matrix $A > 0$ is TP_2 if an only if $A_\pi < A_\sigma$ whenever $\sigma <_{Br} \pi$.

- $(\sigma <_{Br} \pi \Rightarrow A_\pi < A_\sigma) \Rightarrow A$ is TP_2

$$A = \begin{bmatrix} & \vdots & \vdots & \\ \cdots & a_{ij} & a_{i(j+1)} & \cdots \\ \cdots & a_{i+1} & a_{(i+1)(j+1)} & \cdots \\ & \vdots & \vdots & \end{bmatrix}$$

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Thank You!