

How to construct (v, k, λ) -designs fast and easily

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Designs

Design

Design is a pair (X, \mathcal{A}) where

- 1 X is a set of elements called **points**, and
- 2 \mathcal{A} is a multiset of nonempty subsets of X called **blocks**.

 $t - (v, k, \lambda)$ - design

Let t, v, k and λ be positive integers such that $v > k \geq 2$. A $t - (v, k, \lambda)$ -**design** is a design (X, \mathcal{A}) such that the following properties are satisfied :

- 1 $|X| = v$,
- 2 each block contains exactly k points, and
- 3 every set of t distinct points is contained in exactly λ blocks.

(v, k, λ) - design

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- ① $|X| = v$,
 - ② each block contains exactly k points, and
 - ③ every pair of distinct points is contained in exactly λ blocks.
- An example of $(16, 6, 2)$ - design

■	■	■	■
■	■	■	■
■	■	■	■
■	■	■	■

Automorphism group

Design automorphism

Automorphism of a design (X, \mathcal{A}) is a mapping $\pi : X \rightarrow X$ so that if $B \in \mathcal{A}$, then $\pi(B) \in \mathcal{A}$.

It can be shown that the set of all automorphisms of a (X, \mathcal{A}) forms a group G under the operation of composition of mappings.

- Dihedral group of order 10

$$G = \langle a, b \mid a^5 = b^2 = 1, a^b = a^{-1} \rangle$$

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- **Dihedral group of order 10**

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Example (16, 6, 2)

Let G act on (16, 6, 2) in a following way :

- point orbits $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$
- point orbit sizes $\gamma = [1, 5, 5, 5]$

Point orbits

- $\mathcal{O}_1 = \{1\}$
- $\mathcal{O}_2 = \{2, 3, 4, 5, 6\}$
- $\mathcal{O}_3 = \{7, 8, 9, 10, 11\}$
- $\mathcal{O}_4 = \{12, 13, 14, 15, 16\}$

Then we have

$$G = \langle (2, 3, 4, 5, 6)(7, 8, 9, 10, 11)(12, 13, 14, 15, 16), \\ (3, 6)(4, 5)(8, 11)(9, 10)(13, 16)(14, 15) \rangle$$

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Kramer-Mesner

- $G =$ group of permutations of $\{1, 2, \dots, v\}$
- $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m$ t -orbits of $\{1, \dots, v\}$
- $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ k -orbits of $\{1, \dots, v\}$

$$a_{ij} = |\{B \in \mathcal{B}_j \mid S \subseteq B\}|, \quad S \in \mathcal{S}_i \quad (\text{does not depend on choice!})$$

- Kramer-Mesner matrix $A_{tk}^G = [a_{ij}]$

- A_{26}^G

	\mathcal{B}_1	\mathcal{B}_2	\mathcal{B}_3	\mathcal{B}_4	\mathcal{B}_5	\mathcal{B}_6	\mathcal{B}_7	\mathcal{B}_8	...
\mathcal{S}_1	1	8	8	4	8	8	4	6	...
\mathcal{S}_2	0	2	2	1	0	0	0	4	...
\mathcal{S}_3	0	0	0	0	2	2	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...
\mathcal{S}_{18}	0	0	0	0	0	0	0	0	...

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\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...
\mathcal{S}_{18}	0	0	0	0	0	0	0	0	...

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\mathcal{S}_{18}	0	0	0	0	0	0	0	0	...

Theorem

A simple t - (v, k, λ) design with G as a group of automorphisms exists if and only if the system of linear equations

$$A_{tk}^G \cdot x = \lambda u_n$$

has a $\{0, 1\}$ -solution.

Kramer-Mesner algorithm

- 1 calculate t - orbits
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Theorem

If D is a simple t - (v, k, λ) design with G as a group of automorphisms, then D is $s - (v, k, \lambda)$ -design, $1 \leq s \leq t$ and the system of linear equations

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Example 2-(16,6,2)

- calculate 2 - orbits (**18** 2-orbits)
- calculate 6 - orbits (**860** 6-orbits)
- A_{26}^G is of size 18×860

We need to solve

$$A_{26}^G \cdot x = 2 u_{860}$$

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Example (16,6,2)

$$\begin{bmatrix}
 1 & 8 & 8 & 4 & 8 & 8 & 4 & 6 & 3 & 6 & 6 & 6 & \dots \\
 0 & 2 & 2 & 1 & 0 & 0 & 0 & 4 & 2 & 4 & 4 & 2 & 2 & \dots \\
 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\
 1 & 6 & 6 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 4 & 4 & 4 & \dots \\
 1 & 6 & 6 & 3 & 6 & 6 & 3 & 2 & 1 & 2 & 2 & 2 & 2 & \dots \\
 0 & 2 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 2 & 2 & \dots \\
 0 & 1 & 2 & 1 & 0 & 0 & 0 & 3 & 1 & 2 & 2 & 1 & 1 & \dots \\
 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 1 & 1 & \dots \\
 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7 \\
 \vdots \\
 x_{860}
 \end{bmatrix}
 =
 \begin{bmatrix}
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 \vdots \\
 2
 \end{bmatrix}$$

Can we reduce this problem?
YES WE CAN!

Example (16,6,2)

$$\begin{bmatrix}
 1 & 8 & 8 & 4 & 8 & 8 & 4 & 6 & 3 & 6 & 6 & 6 & \dots \\
 0 & 2 & 2 & 1 & 0 & 0 & 0 & 4 & 2 & 4 & 4 & 2 & 2 & \dots \\
 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\
 1 & 6 & 6 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 4 & 4 & 4 & \dots \\
 1 & 6 & 6 & 3 & 6 & 6 & 3 & 2 & 1 & 2 & 2 & 2 & 2 & \dots \\
 0 & 2 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 2 & 2 & \dots \\
 0 & 1 & 2 & 1 & 0 & 0 & 0 & 3 & 1 & 2 & 2 & 1 & 1 & \dots \\
 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 1 & 1 & \dots \\
 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7 \\
 \vdots \\
 x_{860}
 \end{bmatrix}
 =
 \begin{bmatrix}
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 \vdots \\
 2
 \end{bmatrix}$$

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YES WE CAN !

Matrix decomposition

Matrix decomposition

Let A be incidence matrix of a design (X, \mathcal{A}) .

A **decomposition of A** is any partition $\mathcal{O}_1, \dots, \mathcal{O}_r$ of the rows of A (points of (X, \mathcal{A})) and a partition $\mathcal{B}_1, \dots, \mathcal{B}_s$ of the columns of A (blocks of (X, \mathcal{A})).

We say that a decomposition is

- **row-tactical (point-tactical)** if the sum of entries of each row in $\mathcal{O}_i, i = 1, \dots, r$ is constant,
- **column-tactical (block-tactical)** if the sum of entries of each column in $\mathcal{B}_j, j = 1, \dots, s$ is constant.
- **tactical** if it is both row-tactical and column-tactical.

Tactical decomposition matrix (16, 6, 2)

- incidence matrix \Rightarrow tactical decomposition matrix

$$\begin{bmatrix}
 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 \hline
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1
 \end{bmatrix}
 \Rightarrow
 \begin{bmatrix}
 1 & 5 & 0 & 0 \\
 1 & 1 & 2 & 2 \\
 0 & 2 & 3 & 1 \\
 0 & 2 & 1 & 3
 \end{bmatrix}$$

Tactical decomposition matrix $(16, 6, 2)$

- incidence matrix \Rightarrow tactical decomposition matrix

$$\begin{bmatrix}
 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
 \hline
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1
 \end{bmatrix}
 \Rightarrow
 \begin{bmatrix}
 1 & 5 & 0 & 0 \\
 1 & 1 & 2 & 2 \\
 0 & 2 & 3 & 1 \\
 0 & 2 & 1 & 3
 \end{bmatrix}$$

Tactical decomposition matrix

Tactical decomposition matrix

Let \mathcal{P} be tactical decomposition of a design (X, \mathcal{A}) . Number of points from orbit $\mathcal{O}_i, \forall i$ contained in each block in $\mathcal{B}_j, \forall j$ is constant. Matrix containing these values is called **tactical decomposition matrix**.

Lemma 1.

Let (X, \mathcal{A}) be a design, and G an automorphism group of a design (X, \mathcal{A}) .

Then the point and block orbits of G form a tactical decomposition of (X, \mathcal{A}) .

Tactical decomposition

Theorem.

Let D be a (v, k, λ) - design with G as group of automorphism.

- Point orbits : $\mathcal{O}_1, \dots, \mathcal{O}_m$
- Block orbits : $\mathcal{B}_1, \dots, \mathcal{B}_n$
- $\rho_{ij} = |\{B \in \mathcal{B}_j \mid p \in B\}|, p \in \mathcal{O}_i$

Then

$$\sum_{j=1}^n \rho_{ij} = r, \quad 1 \leq i \leq m$$

$$\sum_{i=1}^m \frac{|\mathcal{O}_i|}{|\mathcal{B}_j|} \rho_{ij} = k, \quad 1 \leq j \leq n$$

$$\sum_{j=1}^n \frac{|\mathcal{O}_i|}{|\mathcal{B}_j|} \rho_{ij} \rho_{i'j} = \begin{cases} \lambda |\mathcal{O}_i|, & \text{for } i \neq i' \\ \lambda (|\mathcal{O}_i| - 1) + r, & \text{for } i = i' \end{cases}$$

Example (16, 6, 2)

Tactical decomposition matrix candidate

$$T = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

- Computed with the program *orbmat5qd* developed by V. Krčadinac

Example (16, 6, 2)

 Tactical decomposition matrix candidate

$$T = \begin{array}{c|cccc} \gamma \backslash \beta & 1 & 5 & 5 & 5 \\ \hline 1 & 1 & 5 & 0 & 0 \\ 5 & 1 & 1 & 2 & 2 \\ 5 & 0 & 2 & 3 & 1 \\ 5 & 0 & 2 & 1 & 3 \end{array}$$

- each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.
- each point in \mathcal{O}_2 is contained in exactly 2 blocks in \mathcal{B}_3

Connection : Kramer-Mesner \longleftrightarrow Tactical decomposition

Tactical decomposition

- T ... each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.

$\gamma \backslash \beta$	1	5	5	5
1	1	5	0	0
5	1	1	2	2
5	0	2	3	1
5	0	2	1	3

Kramer-Mesner

- A_{1k}^G ... each point $p \in \mathcal{O}_i$ is contained in exactly a_{ij} blocks $b \in \mathcal{B}_j$

	1	5	5	5	5	5	5	5	5	5	5	5	5	10	10	5	10	...
1	1	5	5	5	5	0	0	0	0	0	0	0	0	10	10	5	10	...
5	1	1	1	1	1	2	2	2	2	2	2	2	2	8	8	4	8	...
5	0	2	2	2	2	3	3	3	3	1	1	1	1	2	2	1	0	...
5	0	2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	2	...

Connection : Kramer-Mesner \longleftrightarrow Tactical decomposition

Tactical decomposition

- T ... each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.

$\gamma \backslash \beta$	1	5	5	5
1	1	5	0	0
5	1	1	2	2
5	0	2	3	1
5	0	2	1	3

Kramer-Mesner

- A_{1k}^G ... each point $p \in \mathcal{O}_i$ is contained in exactly a_{ij} blocks $b \in \mathcal{B}_j$

	1	5	5	5	5	5	5	5	5	5	5	5	5	10	10	5	10	...
1	1	5	5	5	5	0	0	0	0	0	0	0	0	10	10	5	10	...
5	1	1	1	1	1	2	2	2	2	2	2	2	2	8	8	4	8	...
5	0	2	2	2	2	3	3	3	3	1	1	1	1	2	2	1	0	...
5	0	2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	2	...

Connection : Kramer-Mesner \longleftrightarrow Tactical decomposition

• Tactical decomposition

$$T = \begin{array}{c|cccc} \gamma \backslash \beta & 1 & 5 & 5 & 5 \\ \hline 1 & 1 & 5 & 0 & 0 \\ 5 & 1 & 1 & 2 & 2 \\ 5 & 0 & 2 & 3 & 1 \\ 5 & 0 & 2 & 1 & 3 \end{array}$$

 • Reduced A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3

Example (16, 6, 2)

- **18** 2-orbits
- **13** 6-orbits
- reduced A_{26}^G is of dimension 18×13

Reduction

$$18 \times 860 \rightsquigarrow 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

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- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

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$$18 \times 860 \rightsquigarrow 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

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- **18** 2-orbits
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Reduction

$$18 \times 860 \rightsquigarrow 18 \times 13$$

+ 4 additional equations

- Reduced A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Example (16, 6, 2)

- **18** 2-orbits
- **13** 6-orbits
- reduced A_{26}^G is of dimension 18×13

Reduction

$$18 \times 860 \rightsquigarrow 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3

Reduced Kramer-Mesner matrix A_{26}^G

$$\begin{bmatrix}
 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\
 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\
 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\
 0 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7 \\
 x_8 \\
 x_9 \\
 x_{10} \\
 x_{11} \\
 x_{12} \\
 x_{13}
 \end{bmatrix}
 =
 \begin{bmatrix}
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2 \\
 2
 \end{bmatrix}$$

Example (16, 6, 2)

$$x = [1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0]$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Examples of reduction

v	k	λ	Reduction		
16	6	2	18×860	\rightsquigarrow	18×13
7	3	1	7×13	\rightsquigarrow	7×7
13	5	5	26×430	\rightsquigarrow	26×118
13	3	1	26×99	\rightsquigarrow	26×42
22	7	6	15×8167	\rightsquigarrow	15×457
15	5	10	7×152	\rightsquigarrow	7×88
14	4	6	5×54	\rightsquigarrow	5×26
15	5	4	7×52	\rightsquigarrow	7×42

Results

v	k	λ	Handbook	Found
14	4	6	> 4	> 1091
14	4	15	> 1	> 63974
15	5	8	> 104	> 192
15	5	10	> 1	> 296717
15	6	10	> 118	> 3727
21	6	4	> 1	> 1351
21	6	5	> 1	> 5328
21	6	6	> 1	> 16977
22	6	5	> 3	> 4897
22	7	6	> 1	> 28