Cops and Robber with Road Blocks

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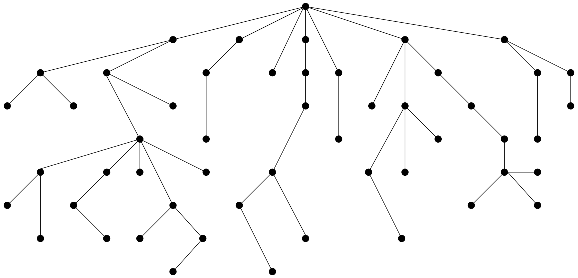
The Game

- There are many versions of this game.
- The key is to define precisely all conditions.
  - eg. how many cops, visibility if any, winning conditions for each player, traps, etc.
- The new element of this game will be road blocks, which will be represented by deleting an edge.

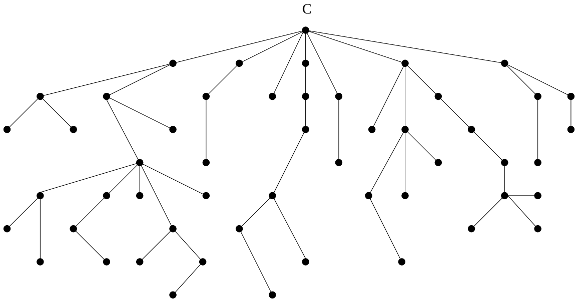
The Rules

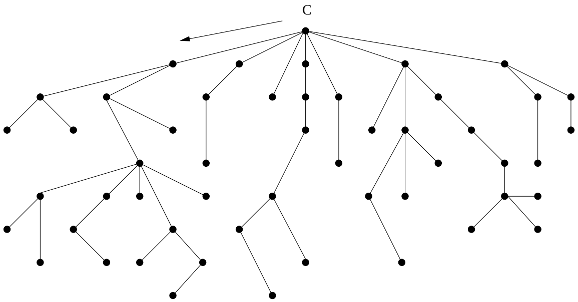
- We play the game on a finite reflexive undirected graph G, with no multiple edges.
- We will use one cop. He will also have a visibility distance of one.
- Once a road block is placed, the robber cannot use that edge but the cop can.
- The game begins with the cop then robber choosing positions on the graph, then each player taking turns at moving until the game is won.
- A move consists of a player either remaining at their current position or moving to an adjacent vertex in the graph.
- The robber will have perfect information while the cop will only have knowledge of the structure of the graph, or will be able to deduce it.
- The cop wins if he can catch the robber, while the robber wins if he can elude the cop forever.

An Example



An Example

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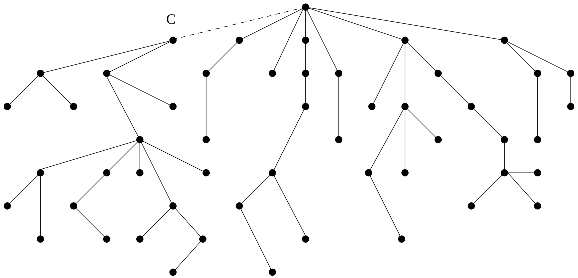
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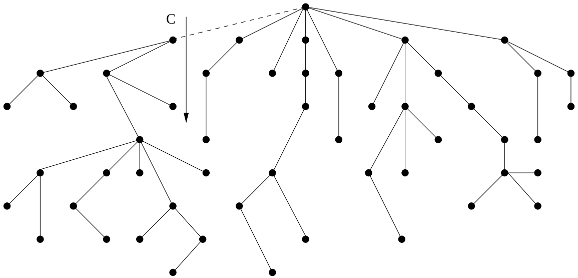
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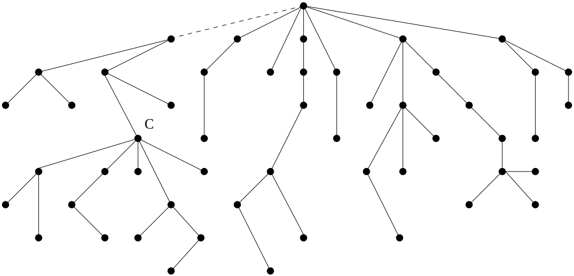
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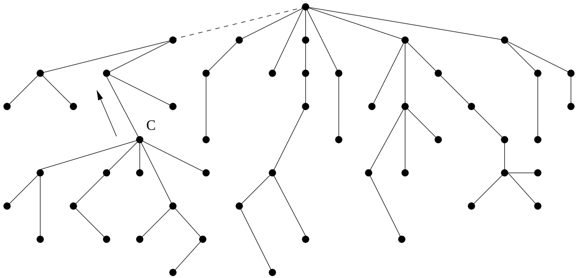
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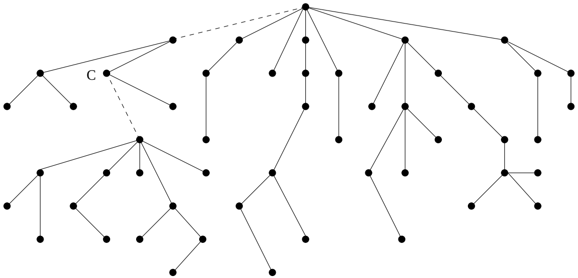
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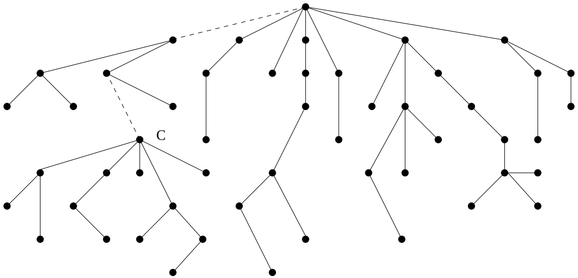
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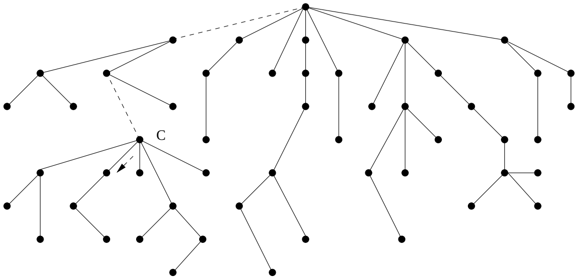
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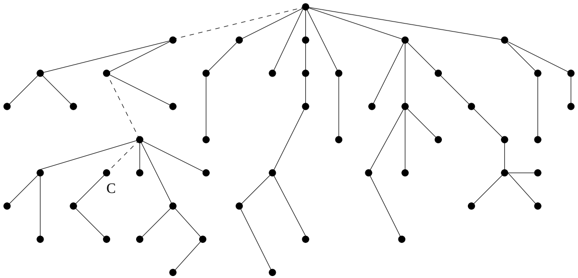
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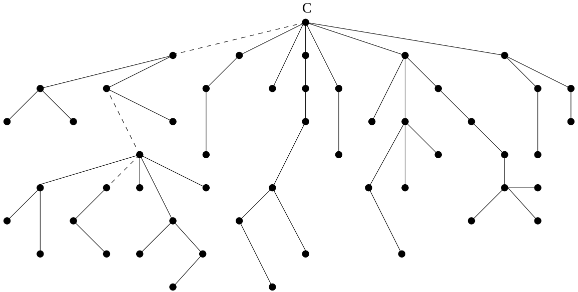
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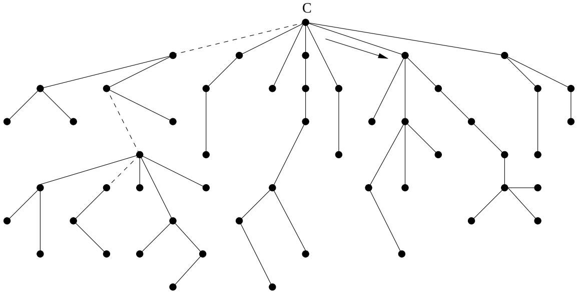
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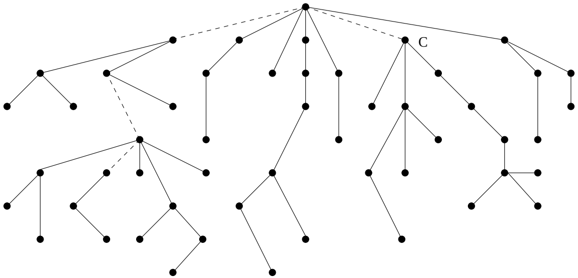
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The Question

Given a graph G, what is the minimum number of road blocks required by the cop to catch the robber?

Definitions

Definition

The road block number of a graph G, denoted as $rb(G)$, is the minimum number of road blocks required by a single cop to catch the robber on G.

Definition

The domination number of a graph G, denoted $\gamma(G)$, is the minimum size of a dominating set of vertices in G.

The Trivial Cases

- There are several cases that can be taken care of immediately under these rules.

Lemma

If $\gamma(G) = 1$ *then* $rb(G) = 0$.

- Paths are a special case which requires no road blocks.
- Cycles are also another special case. Let C_n be a cycle with $n > 3$. Then deletion of any edge in the graph results in a graph isomorphic to P_{n-1} a path of length $n-1$. Since Paths require no road blocks, then original graph C_n requires just one road block.

Let M be the following graph.

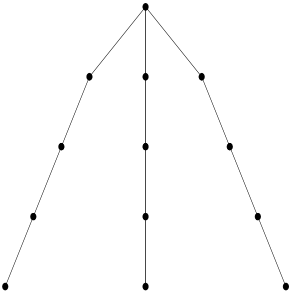


Figure: The Forbidden Tree.

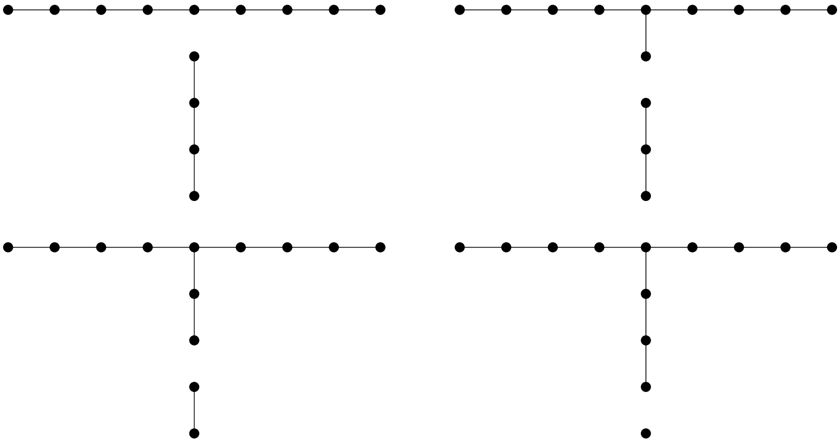


Figure: The Possible Isomorphic Graphs after deleting any edge of M.

Three Theorems

Theorem

Let M be the graph depicted in Figure 1. Then $rb(M) = 1$.

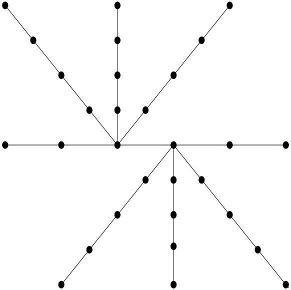
Theorem

Let T be a tree. $rb(T) = 0$ if and only if T is M-free, where M is the graph depicted in Figure 1.

Theorem

If there exists k disjoint instances of the graph M from Figure 1 in a given tree T, then $rb(T) \geq k$.

Notice that the converse to this last theorem is false. We cannot guarantee that the robber will be caught using only k road blocks. As an example, let G be the graph depicted below.



A Classification of One Road Block Trees

Theorem

Let T be a tree and $\mathcal{S}$ be the set of subsets of vertices that induces the graph M from Figure 1. Then $rb(T) = 1$ if and only if $\mathcal{S}$ is non-empty and $\bigcap_{H \in \mathcal{S}} E(H) \neq \emptyset$.

In English: all subgraphs of T isomorphic to M have at least one edge in common.

What about Arbitrary Graphs?

Theorem

1. *Let G be a graph containing at least one cycle with girth at least four. Then $rb(G) \geq 1$.*
2. *Let G be a triangle-free graph with n vertices and m edges. Then $rb(G) \geq m - n + 1$.*

Theorem

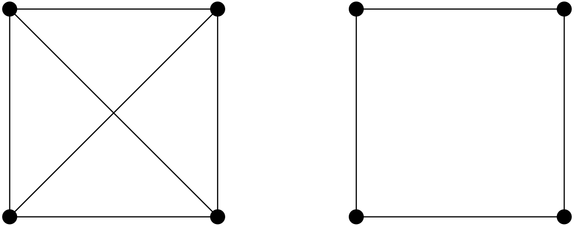
Let G be a triangle-free graph with n vertices and m edges. Further suppose that G contains a spanning tree which is M-free. Then $rb(G) = m - n + 1$.

Subgraphs of a given Graph

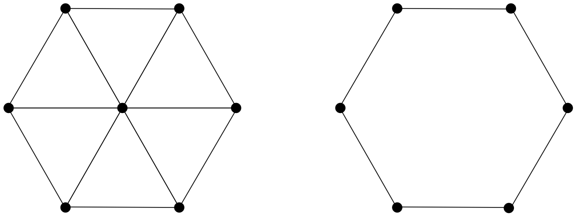
Theorem

Let H be a connected induced isometric subgraph of G. Then $rb(H) \leq rb(G)$.

Subgraph, but not Induced



Induced but not Isometric



Results for Complete Bipartite Grpahs

Theorem

Let $K_{m,n}$ *be a complete bipartite graph with partitions size* m, n *respectively. Then* $rb(K_{m,n}) = mn - m - n + 1 = (m-1)(n-1)$.

Open Questions

- Can we find a searching algorithm for trees?
- Can we play the game on spanning trees?
- What are other forbidden subgraphs to ensures that the road block number of the graph are bounded above a certain number?
  - For example, M is a forbidden subgraph for T if $rb(T) = 0$.
  - For example, given positive integer k, what are forbidden subgraphs for a graph G if $rb(G) \leq k$.
- Application to network searching?
- Moving Road blocks?

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