Pentangulated Graphs
(and Constrained Chords)

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CanaDAM 2009 Université de Montréal May 27, 2009

Theorem 1: Each of the following is equivalent to a graph being chordal:

(1.1) $|C| \geq 4 \Rightarrow C$ has a chord.

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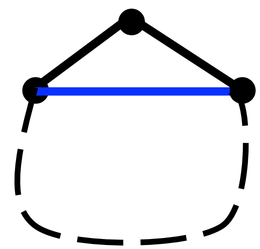
(1.1) $|C| \geq 4 \Rightarrow C$ has a 2-chord.

(a "triangular chord")



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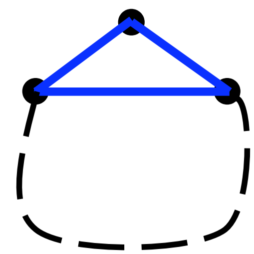
i.e., a 3-cycle that consists of an Edge of *C*, followed by a Chord of *C*, followed by an Edge of *C*



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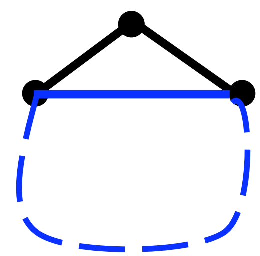
(1.1) $|C| \geq 4 \Rightarrow C$ spans an ECE-cycle.

(1.2) $|C| \geq 4 \Rightarrow C$ is the sum of a triangle and a $(|C| - 1)$-cycle.

i.e., sum in the cycle space

i.e., symmetric difference (as sets of edges)



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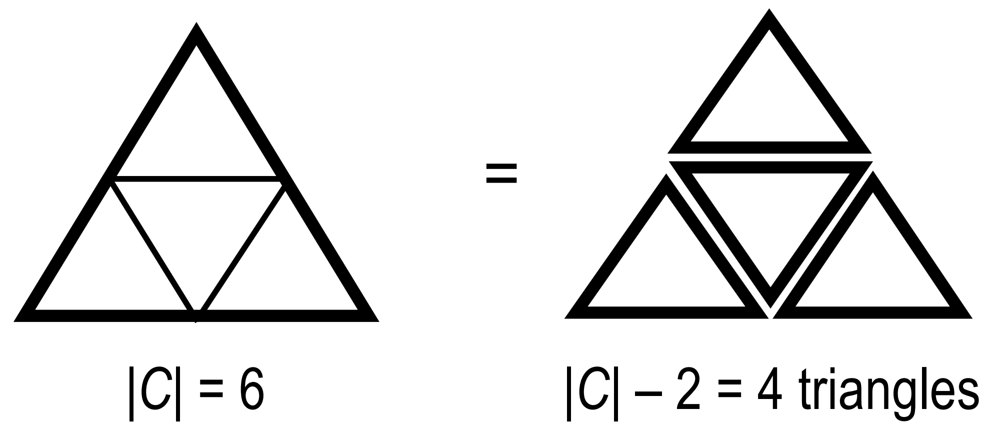

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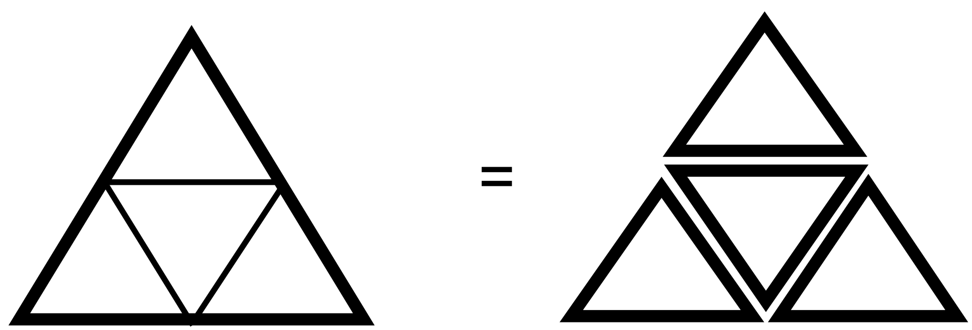


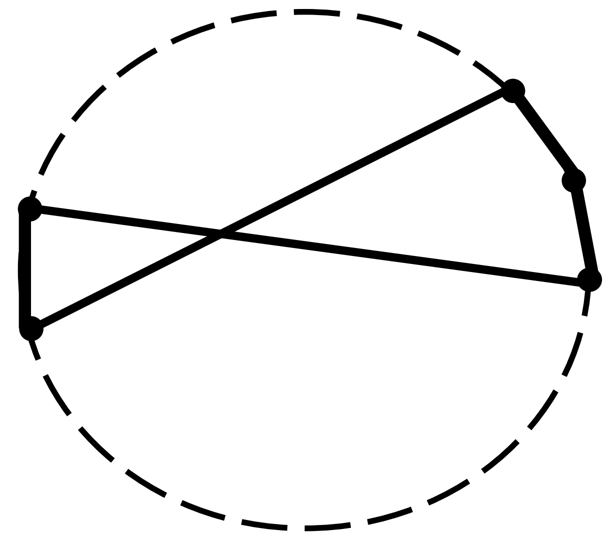
$|C| = 6$ $\quad$ $|C| - 2 = 4$ triangles

chordal ≡ *triangulated*

Theorem 2: Each of the following is equivalent to a graph being incrementally pentangulated:

(2.1) $|C| \geq 6 \Rightarrow C$ spans a crossed ECECE-cycle.

i.e., a 5-cycle that consists of an Edge of *C*, followed by a Chord of *C*, followed by an Edge of *C*, followed by a Chord of *C*, followed by an Edge of *C*



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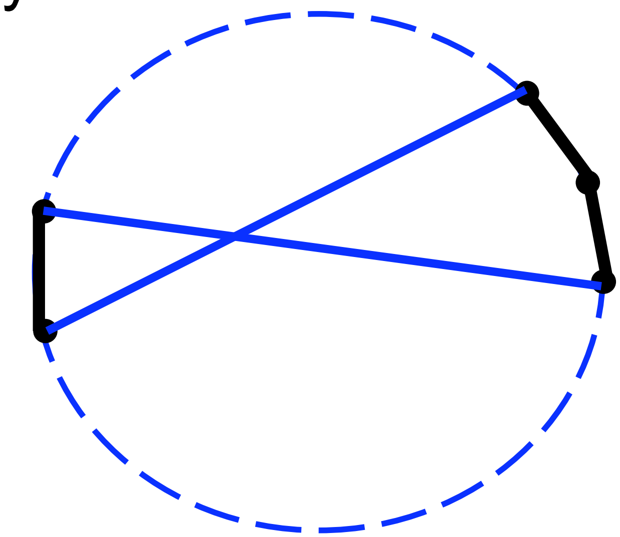

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Definition: A graph is pentangulated if every cycle C with $|C| \geq 6$ is the sum of $|C| - 4$ distinct pentagons spanned by C.

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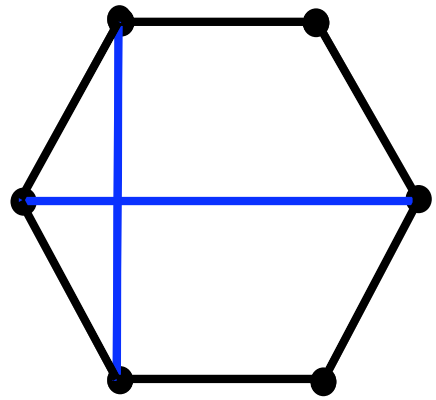
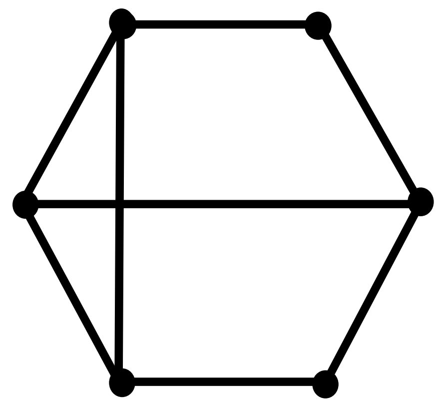
***Conjecture*:** *A graph is pentangulated if and only if it is incrementally pentangulated.*

[verified through order 9 and for …]



An order 6 graph is pentangulated
iff
every 6-cycle has a 2-chord that crosses a 3-chord.



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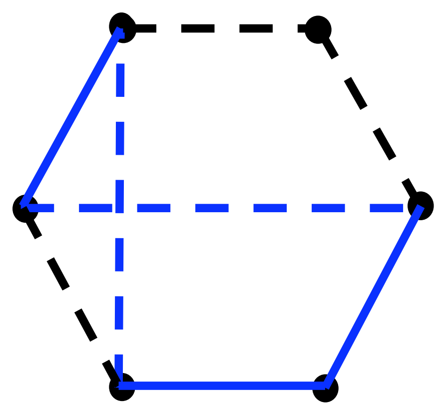
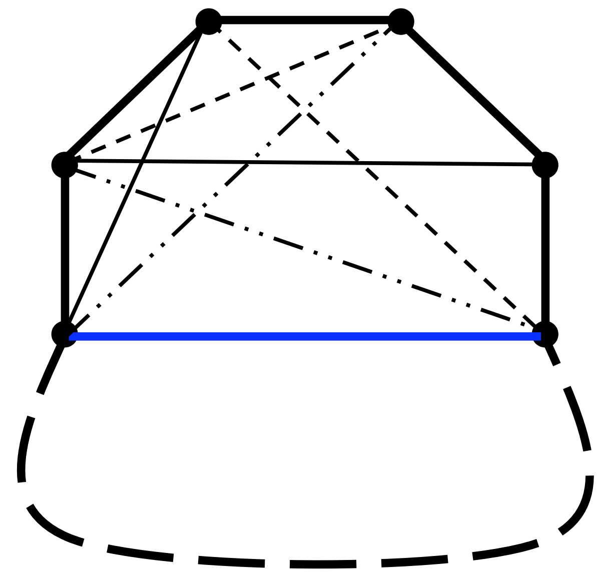


If V(C) induces a pentangulated graph and if C has a 5-chord*, then C spans an ECECE-cycle.



*or a ($|C|$ - 5)-chord if $|C| \leq 9$

If V(*C*) induces a pentangulated graph and if *C* has a 5-chord*, then *C* spans an ECECE-cycle.

Does pentangulated and $|C| \geq 8 \Rightarrow$ *C has a 5-chord**?

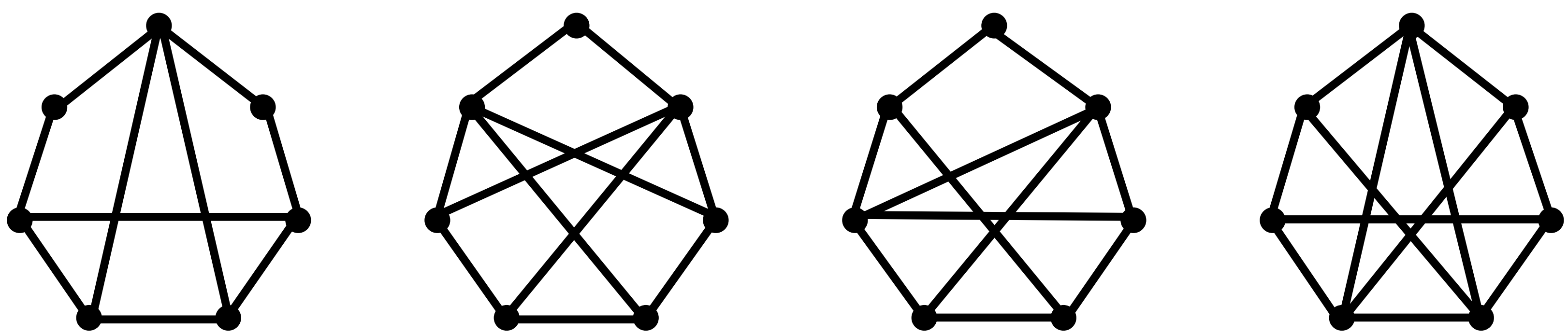
If so, then pentangulated ⇔ incrementally pentangulated.

*or a (|*C*| - 5)-chord if |*C*| ≤ 9

If V(*C*) induces a pentangulated graph and if *C* has a 5-chord, then *C* spans an ECECE-cycle.

Does pentangulated and |*C*| ≥ 8 ⇒ *C has a 5-chord**?

The pentangulated graphs on 7-cycles without (7–5)-chords:



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***Conjecture*:** *A graph is pentangulated if and only if it is incrementally pentangulated.*

What graph class comes in between triangulated graphs and pentangulated graphs?

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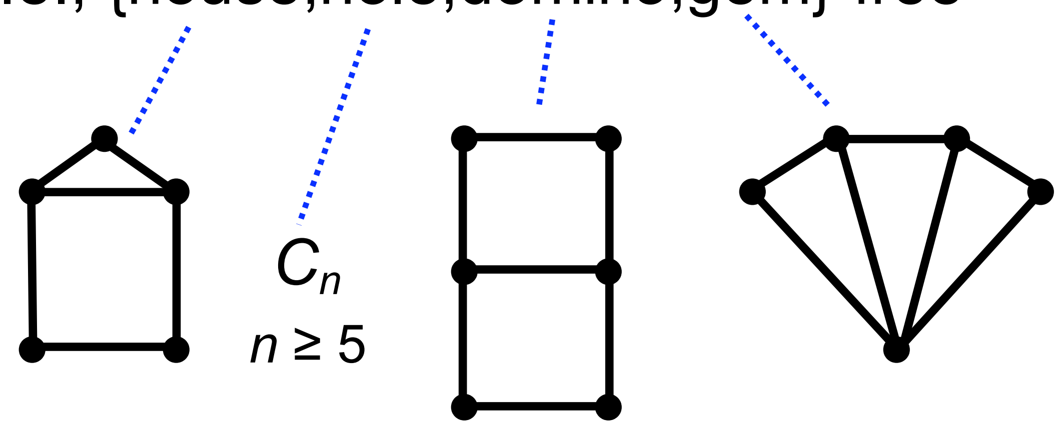
Distance-hereditary graphs??

Theorem 3: Each of the following is equivalent to a graph being distance-hereditary:

i.e., the distance between vertices in a connected induced subgraph of G always equals their distance in G

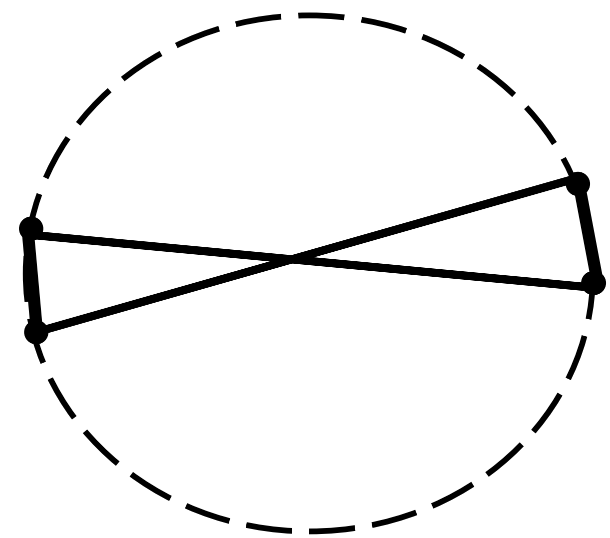
i.e., $|C| \geq 5 \Rightarrow C$ has crossing chords

i.e., {house,hole,domino,gem}-free



Theorem 3: Each of the following is equivalent to a graph being distance-hereditary:

(3.1) $|C| \geq 5 \Rightarrow C$ spans a crossed ECEC-cycle.



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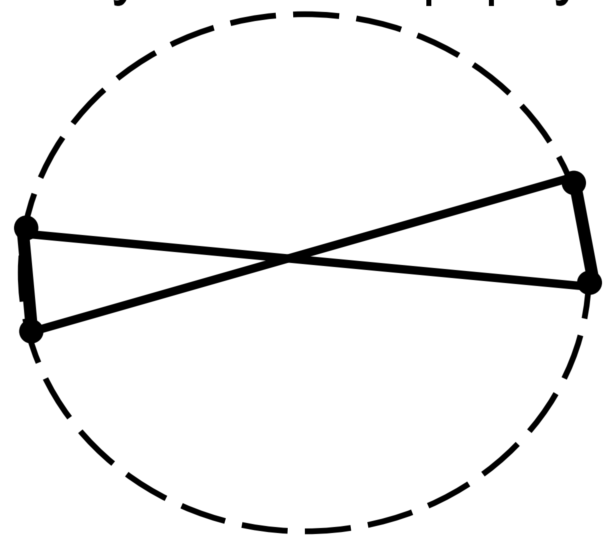

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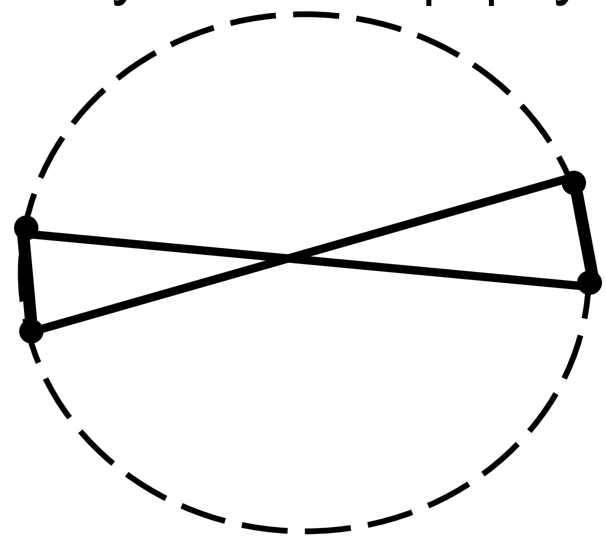


Why not $|C| \geq 4 \Rightarrow$?

Theorem 4: Each of the following is equivalent to a graph being ———:

(4.1) $|C| \geq 5 \Rightarrow C$ spans a crossed ECECE-cycle.

(4.2) ———

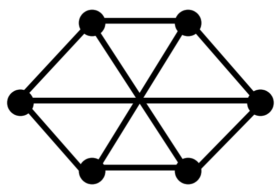
(4.3) ———

Theorem 4: Each of the following is equivalent to a graph having $\{P_4, 2K_2, K_{3,3}, K_{2,2,2}\}$-free blocks:

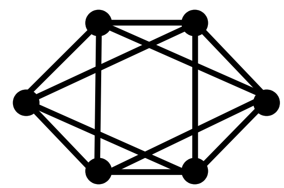
(4.1) $|C| \geq 5 \Rightarrow C$ spans a crossed ECECE-cycle.

(4.2) ———

(4.3) ———



$K_{3,3}$



$K_{2,2,2}$

Theorem 4: Each of the following is equivalent to a graph having $\{P_4, 2K_2, K_{3,3}, K_{2,2,2}\}$-free blocks:

(4.1) $|C| \geq 5 \Rightarrow C$ spans a crossed ECECE-cycle.

(4.2) G is both distance-hereditary and incrementally pentangulated.

(4.3) G is both distance-hereditary and pentangulated.

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Corollary: Pentangulated $\Leftrightarrow$ incrementally pentangulated for all distance-hereditary graphs.

Pentangulated Graphs
(*and Constrained Chords*)

Theorem A: Each of the following is equivalent to a graph being chordal:

(A.1) ———

(A.2) $|C| \geq 4 \Rightarrow C$ has a 2-chord.

Theorem B: Each of the following is equivalent to a graph being strongly chordal:

(B.1) chordal and $|C| \geq 6 \Rightarrow C$ has an odd chord.

(B.2) ———

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(A.1) *"Long enough cycles always have even chords,*

(B.1) *and long enough cycles always have odd chords."*

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(A.1) *"Long enough cycles always have even chords,*
(B.1) *and long enough cycles always have odd chords."*

(A.2) *"Long enough cycles always have 2-chords,*
(B.2) *and long enough cycles always have 3-chords."*

Theorem C: Each of the following is equivalent to a graph being distance-hereditary:

(C.1) $|C| \geq 5 \Rightarrow C$ has crossing chords with an even sum.

the sum of crossing k_1- and k_2-chords equals $k_1 + k_2$

i.e.,

"C has crossing chords of same parity"

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(C.2) $|C| \geq 5 \Rightarrow C$ has crossing 2-chords *or* crossing 3-chords.

— *RECALL* —

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Theorem D: Each of the following is equivalent to a graph being ——— :

(D.1) distance-hereditary and $|C| \geq 6 \Rightarrow C$ has crossing chords with an odd sum.

(D.2) distance-hereditary and $|C| \geq 6 \Rightarrow C$ has ———

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(4.5) $|C| \geq 6 \Rightarrow C$ spans a crossed ECECE-cycle.

(4.7) G is both distance-hereditary and pentangulated.

~~Conclusion~~

Conjecture

Pentangulated $\Leftrightarrow$ *incrementally pentangulated.*

Conjecture

Pentangulated ⇔ incrementally pentangulated.

In other words, the equivalence of the following:

$|C| \geq 6 \Rightarrow C$ is the sum of a pentagon and a $(|C| - 1)$-cycle.

$|C| \geq 6 \Rightarrow C$ is the sum of $|C| - 4$ distinct pentagons spanned by C.