A New Characterization of König-Egervary Graphs

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CanaDAM
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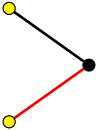
The König-Egervary Theorem: For a bipartite graph, the sum of the independence number α of the graph and its matching number μ equals its number of vertices n (i.e. $\alpha + \mu = n$).

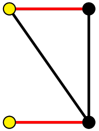


$\alpha = 2$,

$\mu = 1$, and

$n = 3$

A König-Egervary graph (or KE graph) is a graph where the sum of the independence number α of the graph and its matching number μ equals its number of vertices n (i.e. $\alpha + \mu = n$).



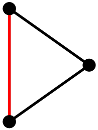
$\alpha = 2$,

$\mu = 2$, and

$n = 4$

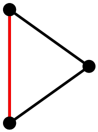
Deming's Characterization (1979)

- Start with a maximum matching M in a graph G.



Deming's Characterization (1979)

- It can be assumed that the matching M is perfect.
- If not, extend G to a graph G'.



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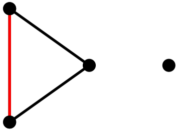

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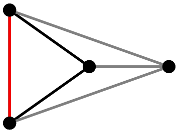

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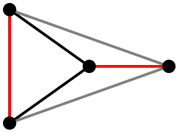

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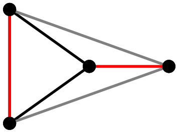
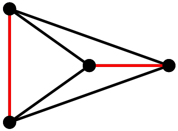


- G is KE iff its extension G' is KE.

Deming's Characterization (1979)

- Find all blossoms and blossom tips with respect to the matching M.



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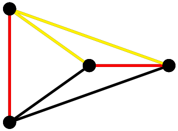

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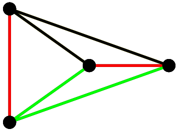
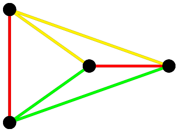


Deming's Characterization (1979)

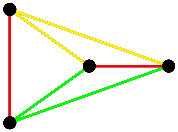
- A blossom pair is a pair of blossoms whose blossom tips are joined by an M-alternating path beginning and ending with edges in M.



Deming's Characterization (1979)

Theorem

A graph G with a perfect matching M is KE iff G contains no blossom pair.



$\alpha = 1$

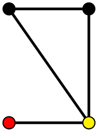
$\mu = 2$

$n = 4$

A New Characterization of KE Graphs

- This is in terms of critical independent sets (Zhang, 1990).

The critical difference d is the maximum value of $|I| - |N(I)|$, for all independent sets I. An independent set I_c which realizes d is a critical independent set.



Let I_c=red vertices,

then $N(I_c)$=yellow vertices,

and $|I| - |N(I)| = 0$.

$d = 0$ and I_c is a critical independent set.

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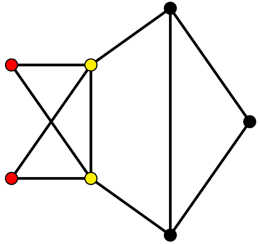


Let I_c=red vertices,

then $N(I_c)$=yellow vertices,

and $|I_c| - |N(I_c)| = 0$.

$d = 0$ and I_c is a critical independent set.

The Significance of Critical Independent Sets

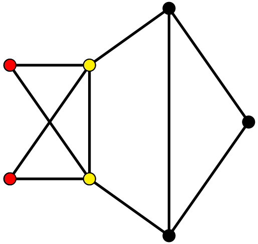
Theorem

(Butenko & Trukhanov, 2007) Any critical independent set can be extended to a maximum independent set.

The Matching Lemma (I.)

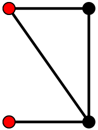
Lemma

For any critical independent set I_c, there is a matching from $N(I_c)$ into I_c.



- This follows from Hall's Theorem.

The critical independence number α' is the cardinality of a maximum independent set which realizes the critical difference d. If I_c is a maximum cardinality critical independent set then $\alpha' = |I_c|$.



Let I_c=red vertices,

I_c is a maximum cardinality critical independent set.

So, $\alpha' = 2$.

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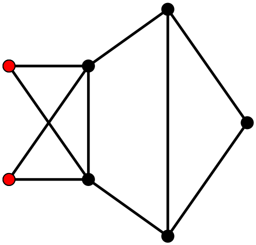
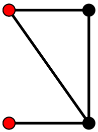


Let I_c=red vertices,

I_c is a maximum cardinality critical independent set.

So, $\alpha' = 2$.

A graph is totally independence reducible if $\alpha' = \alpha$.



Let I_c=red vertices,

I_c is a maximum cardinality critical independent set

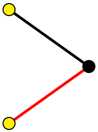
and a maximum independent set.

So, $\alpha' = \alpha$.

For what graphs does $\alpha' = \alpha$?

- DeLaVina asked Graffiti.pc...

Graffiti.pc's Conjecture: $\alpha' = \alpha$ iff G is a KE-graph.



$$\alpha = \alpha' = 2,$$
$$\mu = 1,\ n = 3, \text{ and}$$
$$\alpha + \mu = n$$

Theorem: $\alpha' = \alpha$ iff G is a KE-graph.

Proof.

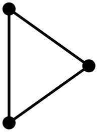
- Suppose $\alpha' = \alpha$.
- Let I be a maximum critical independent set (and a maximum independent set).
- There is a matching from $N(I)$ into I.
- Clearly, $\mu = |N(I)|$.
- So, $\alpha + \mu = |I| + |N(I)| = n$.

Theorem: $\alpha' = \alpha$ iff G is a KE-graph.

- Now suppose G is a KE-graph.
- Let I_c be a maximum critical independent set, and I be a maximum independent set such that $I_c \subseteq I$.
- G is KE implies there is a matching from $N(I)$ into I and, in particular, from $N(I) \setminus N(I_c)$ into $I \setminus I_c$.
- So $|I \setminus I_c| \geq |N(I) \setminus N(I_c)|$.
- But $|I| - |N(I)| = |I \setminus I_c| + |I_c| - (|N(I) \setminus N(I_c)| + |N(I_c)|) = (|I_C| - |N(I_c)|) + (|I \setminus I_c| - |N(I) \setminus N(I_c)|)$.
- So $I = I_c$, and $\alpha' = \alpha$.

A graph is independence irreducible if $\alpha' = 0$.

- For these graphs, for every independent set I, $|N(I)| > |I|$.



$I_c = \emptyset$ is the unique critical independent set,

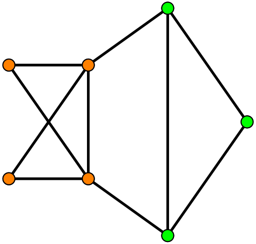
So, $\alpha' = |I_c| = 0$.

An Independence Decomposition

Theorem: For any graph G, there is a unique set $X \subseteq V(G)$ such that

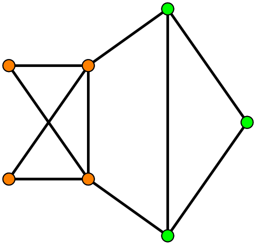
1. $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
2. $G[X]$ is totally independence reducible ($\alpha' = \alpha$),
3. $G[X^c]$ is independence irreducible ($\alpha' = 0$), and
4. for every maximum critical independent set J_c of G, $X = J_c \cup N(J_c)$.

An Independence Decomposition



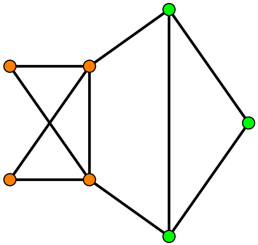
- X is orange, X^c is green,
- $G[X]$ is totally reducible ($\alpha' = \alpha$), and
- $G[X^c]$ is independence irreducible ($\alpha' = 0$).

An Independence Decomposition



- $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c]) = 3.$
- Every graph decomposes into a KE graph and a graph where every independent set I has more than $|I|$ neighbors.

A Question: For what graphs does $\alpha = \mu$?



$\alpha = \mu = 3.$

A Matching Decomposition

Corollary

If J_c is a maximum critical independent set of G and $X = J_c \cup N(J_c)$, then

$$\mu(G) = \mu(G[X]) + \mu(G[X^c]).$$

- Every graph can be decomposed into a KE graph and a graph where, for every independent et I, I has more than $|I|$ neighbors.
- For KE graphs, $\alpha \geq \mu$.

The Matching Lemma (II.)

Lemma

If G is independence irreducible ($\alpha' = 0$), and I is any independent set, them there is a matching from I into N(I).

- This follows from Hall's Theorem.
- For independence irreducible graphs, $\alpha \leq \mu$.

A Hint

Given the independence decomposition $V(G) = X \cup X^c$,

- $\alpha(G[X]) \geq \mu(G[X])$,
- $\alpha(G[X^c]) \leq \mu(G[X^c])$,
- $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
- $\mu(G) = \mu(G[X]) + \mu(G[X^c])$.

So, $\alpha = \mu$ implies $\alpha(G[X]) - \mu(G[X]) = \mu(G[X^c]) - \alpha(G[X^c])$.

Selected References

- E. DeLaVina, Written on the Wall II, Conjectures of Graffiti.pc, at: http://cms.dt.uh.edu/faculty/delavinae/research/wowII/
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- C.-Q. Zhang, Finding critical independent sets and critical vertex subsets are polynomial problems, *SIAM J. Discrete Mathematics*, 3:431–438, 1990.
- S. Butenko and S. Trukhanov, Using Critical Sets to Solve the Maximum Independent Set Problem, *Operations Research Letters* 35(4) (2007) 519–524.
- C. E. Larson, A Note on Critical Independent Sets, *Bulletin of the Institute of Combinatorics and its Applications*, Sept. 2007.
- C. E. Larson, The Critical Independence Number and an Independence Decomposition, submitted.