Combinatorial properties of f-palindromes

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25 mai 2009

Outline

Aims of the talk

- Hof, Knill and Simon conjectured in 1995 a characterization of the fixed point of morphisms having an infinite palindrome complexity (the number of palindrome factors).
- Recently, this conjecture was solved for the binary alphabet (Tan, 2007).
- We show a similar result for fixed points of uniform morphisms having an infinite number of f-palindromes.

Definitions and notations

- A set Σ called alphabet whose elements are called letters.
- Elements w of the free monoid Σ^* are called words. We note $w \in \Sigma^*$ and

$$w = w_0 w_1 w_2 \cdots w_{n-1}, w_i \in \Sigma.$$

- The length of w is $|w| = n$.
- An infinite word $w = w_0 w_1 \cdots$ is a map $w : \mathbb{N} \to \Sigma$.
- If $w = pfs$, then p is called a prefix, f a factor and s a suffix of w.
- $\mathrm{Fact}(w)$ is the set of the (finite) factors of w.

Definitions and notations

- The reversal of a finite word w

$$\widetilde{w} = w_{n-1} w_{n-2} \cdots w_1 w_0 .$$

- A palindrome is a word w such that $w = \widetilde{w}$.
- $\mathrm{Pal}(w) = \mathrm{Fact}(w) \cap \mathrm{Pal}(\Sigma^*)$ is the set of the palindrome factors of w.

Definitions and notations

- A morphism is a function $\varphi : \Sigma^* \to \Sigma^*$ such that

$$\varphi(uv) = \varphi(u)\varphi(v) \quad \text{for all} \quad u, v \in \Sigma^*.$$

- A morphism φ is primitive if there exists $k \in \mathbb{N}$ such that every letters of Σ appear in $\varphi^k(\alpha)$ for all $\alpha \in \Sigma$.
- A morphism is uniform if $|\varphi(\alpha)| = |\varphi(\beta)|$ for all $\alpha, \beta \in \Sigma$.
- We denote by $\widetilde{\varphi}$ the morphism defined by $\alpha \mapsto \widetilde{\varphi(\alpha)}$.
- A fixed point of a morphism φ is a word w such that $\varphi(w) = w$.
- We say that φ is a right-conjugate of φ' if there exists a ord $u \in \Sigma^*$ such that

$$\varphi(\alpha)u = u\varphi'(\alpha), \quad \text{for all } \alpha \in \Sigma.$$

Definitions and notations

Example

The non primitive morphism defined on $\Sigma = \{a, b, c, d, e\}$ by $a \mapsto ab, b \mapsto ba, c \mapsto cd, d \mapsto c, e \mapsto e$ has two finite fixed points :

- ε, the empty word
- e

and three infinite fixed points :

- $abbabaabbaababbaabba\cdots$
- $baababbaabbabaabbaab\cdots$
- $cdccdcdccdccdcdccdcdccdccdcdccdccdcdccdc\cdots$

Each fixed point may be obtained by considering

$$\lim_{n\to\infty} \varphi^n(\alpha), \alpha \in \Sigma.$$

About palindrome complexity

Proposition (Droubay, Justin, Pirillo, 2001)

Let w be a finite word. Then,

$$|\mathrm{Pal}(w)| \leq |w| + 1$$

and Sturmian words reach that bound.

Definition (Brlek, Hamel, Nivat, Reutenauer, 2004)

Let w be a finite word. The defect $D(w)$ of w is

$$D(w) = |w| + 1 - |\mathrm{Pal}(w)|.$$

and w is full if $D(w) = 0$. Moreover, the defect of a infinite word is the supremum of the defect of its finite prefixes.

Full words are also called rich in the recent litterature.

Hof, Knill and Simon Conjecture

Definition (Hof, Knill and Simon, 1995)

A morphism φ is in class $\mathcal{P}$ if there exists a palindrome p and for each $\alpha \in \Sigma$ there exists a palindrome q_α such that $\varphi(\alpha) = pq_\alpha$.

Hof, Knill and Simon Conjecture

The morphism

$$\begin{array}{rcl} \varphi: \{a,b\}^* & \rightarrow & \{a,b\}^* \\ a & \mapsto & bb \cdot aba \\ b & \mapsto & bb \cdot a \end{array}$$

is in class $\mathcal{P}$. It has only one fixed point beginning by letter b.

i	0	1	2	3	4	5	6	7
$\lvert\varphi^i(a)\rvert$	1	5	19	71	265	989	3691	13775
$\lvert\mathrm{Pal}(\varphi^i(a))\rvert$	2	6	20	72	266	990	3692	13776

Hof, Knill and Simon Conjecture

The square of the Thue-Morse morphism $\mu : a \mapsto ab, b \mapsto ba$ is in class $\mathcal{P}$:

$$\begin{array}{rccc} \mu^2 : & \{a,b\}^* & \rightarrow & \{a,b\}^* \\ & a & \mapsto & abba \\ & b & \mapsto & baab \end{array}$$

The palindrome complexity table of one of its fixed point is :

i	0	1	2	3	4	5	6	7
$\lvert\mu^i(a)\rvert$	1	2	4	8	16	32	64	128
$\lvert\mathrm{Pal}(\mu^i(a))\rvert$	2	3	5	9	15	29	53	109

Hof, Knill and Simon Conjecture

The morphism

$$\begin{array}{rccc} \varphi : & \{a,b\}^* & \rightarrow & \{a,b\}^* \\ & a & \mapsto & abb \\ & b & \mapsto & ba \end{array}$$

is not in class $\mathcal{P}$. It has two infinite fixed points having both 23 palindromes :

i	0	1	2	3	4	5	6	7	8
$\lvert \mathrm{Pal}(\varphi^i(a)) \rvert$	2	4	8	15	23	23	23	23	23
$\lvert \mathrm{Pal}(\varphi^i(b)) \rvert$	2	3	6	13	18	23	23	23	23

Hof, Knill and Simon Conjecture

In their article, Hof, Knill and Simon also said :

> *"Clearly, we could include into class $\mathcal{P}$ substitutions of the form $s(\alpha) = q_\alpha p$. We do not know whether all palindromic x_s arise from substitutions that are in this extended class $\mathcal{P}$."*

Their quote is now called HKS Conjecture and it may be stated in the folowing way :

Conjecture (Hof, Knill, Simon, 1995)

Let w be a fixed point of a primitive morphism. Then, $|\mathrm{Pal}(w)| = \infty$ if and only if there exists a morphism φ such that $\varphi(w) = w$ and such that either φ or $\widetilde{\varphi}$ is in class $\mathcal{P}$.

Hof, Knill and Simon Conjecture

Proposition (Blondin-Massé, 2007)

The morphism φ defined by $a \mapsto abbab, b \mapsto abb$ is such that neither φ nor $\widetilde{\varphi}$ are in class $\mathcal{P}$ but $\lim_{n\to\infty} \varphi^n(a)$ has an infinite number of palindromes.

Hence, HKS Conjecture must be restated :

Conjecture

Let w be a fixed point of a primitive morphism. Then, $|\mathrm{Pal}(w)| = \infty$ if and only if there exists a morphism φ such that $\varphi(w) = w$ and such that φ has a conjugate in class $\mathcal{P}$.

This question was solved recently in the binary case (B. Tan, 2007).

Main Results

First, we obtained a result less general then B. Tan :

Theorem

Let $\Sigma = \{a, b\}$, $\varphi : \Sigma^* \mapsto \Sigma^*$ be a primitive uniform morphism and $w = \varphi(w)$ an fixed point. Then, w contains arbitrarily long palindromes if and only if φ, $\widetilde{\varphi}$ or φ^2 is in class $\mathcal{P}$.

Our approach is making use of f-palindromes. Therefore, we also obtained an interesting and similar result for f-palindromes...

Main Results

Let $f : \Sigma \to \Sigma$ be an involution which extends to a morphism on Σ^*. We say that $w \in \Sigma^*$ is an f-palindrome if $w = f(\widetilde{w})$.

They are also called f-pseudo-palindrome in the litterature (Anne, Zamboni, Zorca, 2005 ; de Luca, De Luca, 2006 ; Halava, Harju, Kärki, Zamboni, 2007).

Example

Let $\Sigma = \{a, b\}$ and E be the involution $a \mapsto b, b \mapsto a$. The words

$$\varepsilon, ab, ba, abab, aabb, baba, bbaa, abbaab, bababa$$

are E-palindromes.

Main Results

Definition

We say that a morphism φ is in class f-$\mathcal{P}$ if there exists an f-palindrome p and for each $\alpha \in \Sigma$ there exists a f-palindrome q_α such that $\varphi(\alpha) = pq_\alpha$.

Our second result is :

Theorem

Let $\Sigma = \{a, b\}$, $\varphi : \Sigma^* \mapsto \Sigma^*$ be a primitive uniform morphism and $w = \varphi(w)$ an fixed point. If w contains arbitrarily long E-palindromes, then either φ, $\widetilde{\varphi}$, $\varphi \circ \mu$ or $\widetilde{\varphi} \circ \mu$ is in class E-$\mathcal{P}$, where μ is the Thue-Morse morphism.

Further work

This talk belongs to a more general project which is to find a complete characterization of the all the fixed points $\mathbf{u}$ of morphism for the four classes that emerge from palindrome complexity $|\mathrm{Pal}(\mathbf{u})|$ and defect $D(\mathbf{u})$.

$\|\mathrm{Pal}(\mathbf{u})\|$	$D(\mathbf{u})$	Examples
∞	0	Sturmian words, Fibonacci word.
∞	$0 < D(\mathbf{u}) < \infty$	$(aababbaabbabaa)^\omega$
∞	∞	Thue-Morse word.
finite	∞	$a \mapsto abb, b \mapsto ba$

Conjecture (Blondin-Massé, Brlek, Labbé, 2008)

Let $\mathbf{u}$ be the fixed point of a primitive morphism φ. If the defect of $\mathbf{u}$ is such that $0 < D(\mathbf{u}) < \infty$, then $\mathbf{u}$ is periodic.

Remerciements et Références..