

$K_{s,t}$ minors in graphs

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Based on joint work with **N. Prince**

Definitions

A graph H is a **minor of a graph** G

if H can be obtained from G by a sequence of **contracting** edges and **deleting** vertices and/or edges.

In this case, we also say that G **has an H minor**.

A graph is **k -chromatic** if its chromatic number is **exactly** k .

Conjectures

Conjecture 1. [Hadwiger] Every k -chromatic graph has a K_k minor.

Conjecture 2. [Woodall–Seymour] Every $(s + t)$ -chromatic graph has a $K_{s,t}$ minor.

Conjecture 3. [Woodall] Every graph with choosability $(s + t)$ has a $K_{s,t}$ minor.

Another approach

Let $D(H)$ denote the minimum D such that every graph G with average degree at least D has an H minor.

[Kostochka–Thomason]: $D(K_k) = \Theta(k\sqrt{\log k})$.

Myers and Thomason estimated $D(H)$ for most graphs H . Their approach worked well for dense and balanced graphs. An example of a sparse unbalanced graph is $K_{s,t}$ for a small s and a large t .

Graph $M(r, s, t)$ consists of r copies of K_{s+t-1} , where each two copies share the same fixed $s - 1$ vertices.

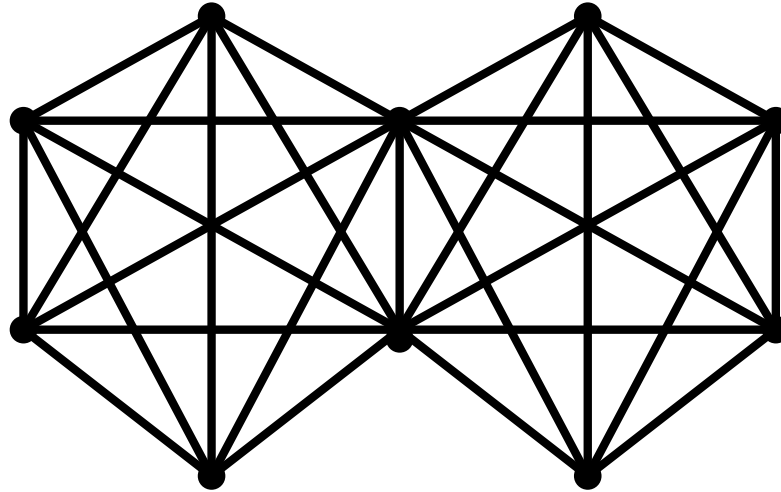


Figure 1: Graph $M(2, 3, 4)$ has no $K_{3,4}$ minor.

$$e(M(r, s, t)) = \frac{1}{2}(t + 2s - 3)(n - s + 1) + \binom{s-1}{2}. \quad (1)$$

Theorem 1. [Myers] Let $t > 10^{29}$ be a positive integer. Let G be a graph with $n \geq 3$ vertices such that

$$e(G) > \frac{1}{2}(t+1)(n-1). \quad (2)$$

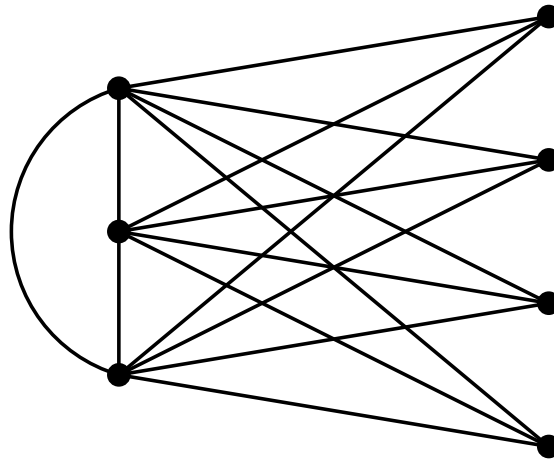
Then G has a $K_{2,t}$ minor.

Theorem 1'. [Chudnovsky–Reed–Seymour] Let G be a graph with $n \geq 3$ vertices satisfying (2). Then G has a $K_{2,t}$ minor.

Corollary. Every $(2+t)$ -chromatic graph has a $K_{2,t}$ minor.

Conjecture 3. [Myers] For each fixed s , there is $C(s)$ such that for all t , $D(K_{s,t}) \leq C(s) \cdot t$.

Let $K_{s,t}^* = K_{s+t} - E(K_t)$.

 $K_{3,4}^*$

Myers: Maybe for fixed s and large t , $D(K_{s,t}^*) = D(K_{s,t})$?

Theorem 2. [Woodall] Every graph with choosability $(2 + t)$ has a $K_{2,t}^*$ minor.

Theorem 3. [Kühn–Osthus] For every $\epsilon > 0$ and every positive integer s , there exists a number $t_0 = t_0(s, \epsilon)$ such that for all $t \geq t_0$, every graph of average degree at least $(1 + \epsilon)t$ has a $K_{s,t}^*$ minor.

Theorem 4. [A. K.–Prince] Let s and t be positive integers with $t > (240s \log_2 s)^{8s \log_2 s + 1}$. Let G be a graph such that

$$e(G) \geq \frac{t + 3s}{2} (n(G) - s + 1).$$

Then G has a $K_{s,t}^*$ minor. Furthermore, for n large, there exists a graph G of order n and size at least $\frac{t + 3s - 5\sqrt{s}}{2} (n - s + 1)$ that has no $K_{s,t}$ minor.

Theorem 5. [A. K.–Prince] Let $t \geq 6500$. Let G be a graph of order $n \geq 3$ with $e(G) > \frac{1}{2}(t+3)(n-2) + 1$. Then G has a $K_{3,t}^*$ minor.

Theorem 6. [A. K.–Prince] Let $s \geq 4$ and $t > 10s$ be such that $s+t$ is odd. Then for infinitely many $n > s+t$, there exists a graph $G(n, s, t)$ of order n with

$$e(G(n, s, t)) > \frac{1}{2}\left(t + 2s - 2 - \frac{2s}{t}\right)(n - s + 1) + \binom{s-1}{2}$$

that has no $K_{s,t}^*$ -minor.

Corollary. [Seymour] Let $t \geq 6500$. Then every $(3+t)$ -chromatic graph has a $K_{3,t}$ minor.

Theorem 7. [A. K.] Let s and t be positive integers such that

$$t > t_0(s) := \max\{4^{15s^2+s}, (240s \log_2 s)^{8s \log_2 s+1}\}. \quad (3)$$

Then every $(s + t)$ -chromatic graph has a $K_{s,t}^*$ minor.

Remark. For every $s, t \geq 3$, there are infinitely many $(s + t)$ -critical graphs that do not have $K_{s,t+1}^*$ minors.

Proof ingredients

Theorem 8. [Gallai] Let $k \geq 3$ and G be a k -critical graph. If $|V(G)| \leq 2k - 2$, then G has a **spanning complete bipartite subgraph**.

Suppose that Theorem 7 is proved for all $s' < s$ and $t' > t_0(s')$. Let G_0 be a minimum w.r.t. $|V(G)| + |E(G)|$ counter-example for s and some $t > t_0(s)$. Then G_0 is $(s + t)$ -critical. By Theorem 8, $|V(G_0)| \geq 2(s + t) - 1$.

Lemma 1. [Seymour] Let $k \geq 0$. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then $\alpha(G_0[N(v)]) \leq k + 1$.

Proof ingredients

Lemma 2. If $t \geq 4^{x+s}$, then the connectivity of G_0 is greater than x .

Lemma 3. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then there exists a subset $Y(v) \subseteq N(v)$ such that

$$\delta(G_0[Y(v) \cup \{v\}]) \geq \frac{t}{k+1}.$$

Lemma 4. Let $t > t_0(s)$. Let G be a $15s^2$ -connected graph. Suppose that G contains a vertex subset U with

$$t + 700s^3 \ln t \leq |U| \leq 3t$$

such that $\delta(G[U]) \geq t/(4s + 1)$. Then G has a $K_{s,t}^*$ -minor.