Repetitions in Words Associated with Parry Numbers

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β-expansion

For each $x \in [0, 1)$ and for each $\beta > 1$, using the greedy algorithm, one can obtain the unique β-expansion $(x_i)_{i \geq 1}$, $x_i \in \mathbb{N}$, $0 \leq x_i < \beta$, of the number x such that

$$x = \sum_{i \geq 1} x_i \beta^{-i} \quad \text{and} \quad \sum_{i \geq k} x_i \beta^{-i} < \beta^{-k+1}.$$

By shifting, each non-negative number has a β-expansion.

Rényi expansion of unity in base $\beta > 1$

$$d_\beta(x) = x_1 x_2 x_3 \cdots , \qquad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \quad x \in [0,1)$$

where

$$T_\beta : [0,1] \to [0,1), \quad T_\beta(z) := \beta z - \lfloor \beta z \rfloor = \{\beta z\}.$$

Rényi expansion of unity in base $\beta > 1$

$$d_\beta(1) = t_1 t_2 t_3 \cdots , \qquad t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor,$$

where

$$T_\beta : [0,1] \to [0,1), \quad T_\beta(z) := \beta z - \lfloor \beta z \rfloor = \{\beta z\}.$$

Definition

- *Parry number: $d_\beta(1)$ is eventually periodic,*
- *simple Parry number: $d_\beta(1) = t_1 \cdots t_m$,*
- *non-simple Parry number: $d_\beta(1) = t_1 \cdots t_m (t_{m+1} t_{m+2} \dots t_{m+p})^\omega$.*

β-integers

Definition

The real number x is a β-integer if the β-expansion of $|x|$ is of the form $\sum_{i=0}^{k} a_i\beta^i$, where $a_i \in \mathbb{N}$. The set of all β-integers is denoted by $\mathbb{Z}_\beta$.

Theorem (Thurston (1989))

The set of lengths of gaps between two consecutive β-integers is finite if and only if β is a Parry number. Moreover, if β is a simple Parry number, i.e., $d_\beta(1) = t_1 \cdots t_m$, the set reads $\{\triangle_0, \triangle_1, \ldots \triangle_{m-1}\}$, if β is a non-simple Parry number, i.e., $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+p})^\omega$, we obtain $\{\triangle_0, \triangle_1, \ldots \triangle_{m+p-1}\}$.

Simple Parry numbers

$d_\beta(1) = t_1 \cdots t_m$

Canonical substitution φ_β over the alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$

$$\begin{array}{rcl} \varphi_\beta(0) & = & 0^{t_1}1 \\ \varphi_\beta(1) & = & 0^{t_2}2 \\ & \vdots & \\ \varphi_\beta(m-2) & = & 0^{t_{m-1}}(m-1) \\ \varphi_\beta(m-1) & = & 0^{t_m} \end{array}$$

Fixed point $\mathbf{u}_\beta = \lim_{n\to\infty} \varphi_\beta^n(0) = 0^{t_1}1\cdots$.

Non-simple Parry numbers

$d_\beta(1) = t_1 \cdots t_m (t_{m+1} t_{m+2} \ldots t_{m+p})^\omega$

Canonical substitution φ_β over the alphabet $\mathcal{A} = \{0, 1, \ldots, m+p-1\}$

$$
\begin{aligned}
\varphi_\beta(0) &= 0^{t_1} 1 \\
\varphi_\beta(1) &= 0^{t_2} 2 \\
&\vdots \\
\varphi_\beta(m-1) &= 0^{t_m} m \\
\varphi_\beta(m) &= 0^{t_{m+1}} (m+1) \\
&\vdots \\
\varphi_\beta(m+p-2) &= 0^{t_{m+p-1}} (m+p-1) \\
\varphi_\beta(m+p-1) &= 0^{t_{m+p}} m
\end{aligned}
$$

Fixed point $\mathbf{u}_\beta = \lim_{n\to\infty} \varphi_\beta^n(0) = 0^{t_1} 1 \cdots$.

Powers of words

Definition

Let w be a nonempty word, $r \in \mathbb{Q}$, then u is r-th power of w if u is a prefix of w^ω and $r = \frac{|u|}{|w|}$, i.e.,

$$u = w^r := w^{\lfloor r \rfloor} w',$$

where w' is a proper prefix of w.

Example

Let $w = 123$ and $v = 12312312312 = (123)^3 12$, then $r = \frac{|v|}{|w|} = \frac{11}{3} = 3 + \frac{2}{3}$ and so v is $\frac{11}{3}$-power of w.

Index of finite words

Definition

Let $\mathbf{u} = (u_i)_{i \geq 1}$ be an infinite word and w its nonempty factor. Then the index of w in $\mathbf{u}$ is given by

$$\mathrm{ind}(w) = \sup\{r \in \mathbb{Q} \mid w^r \text{ is a factor of } \mathbf{u}\}.$$

Example

$\mathbf{u} = 12(121)^\omega = 12\,121\,121\,121\,\cdots$, then

$$\mathrm{ind}(121) = \infty, \mathrm{ind}(12) = 2 + \frac{1}{2}$$

Critical exponent of infinite word

Definition

Let $\mathbf{u} = (u_n)_{n \geq 1}$ be an infinite word. Then the critical exponent of $\mathbf{u}$ is given by

$$\mathrm{E}(\mathbf{u}) = \sup\{\mathrm{ind}(w) \mid w \text{ is a factor of } \mathbf{u}\}.$$

Remark

- $1 < \mathrm{E}(\mathbf{u}) \leq \infty$,
- *Every real number greater than 1 is a critical exponent. [Krieger, Shallit (2007)]*

Quadratic non-simple Parry number

$\beta > 1$ is quadratic non-simple Parry number if $d_\beta(1) = t_1 t_2^\omega$, where $t_1 > t_2 \geq 1$. The word $\mathbf{u}_\beta$ coding the distribution of β-integers on the positive real line is the fixed point of the substitution

$$\varphi_\beta(0) = 0^{t_1}1, \quad \varphi_\beta(1) = 0^{t_2}1, \quad \text{i.e.,} \quad \mathbf{u}_\beta = \lim_{n\to\infty} \varphi_\beta^n(0).$$

$$\mathrm{E}(\mathbf{u}_\beta) = ??$$

Result of Dalia Krieger

Theorem

Let φ be a non-erasing substitution defined over a finite alphabet $\mathcal{A}$, $\mathbf{u} = \varphi^\omega(0)$. Let M_φ be the incidence matrix of φ and $\lambda, \lambda_1, \ldots, \lambda_\ell$ be its eigenvalues. Suppose $\mathrm{E}(\mathbf{u}) < \infty$. Then

$$\mathrm{E}(\mathbf{u}) \in \mathbb{Q}[\lambda, \lambda_1, \ldots, \lambda_\ell].$$

Maximal powers and bispecial factors (1/2)

Definition

A factor v of an binary infinite word $\mathbf{u}$ is bispecial if letters 0 and 1 are both left and right extensions of v, i.e., $0v$, $1v$, $v0$, $v1$ are all factors of $\mathbf{u}$.

Definition

A word $\bar{w}$ is conjugate of a word w if there exists a prefix w' of w such that $\bar{w} = (w')^{-1}ww'$.

Example

$w = avb$, then both vba and bav are conjugates of w.

Maximal powers and bispecial factors

Lemma

Let w be a factor of an infinite binary word $\mathbf{u}$ such that $\infty > ind(w) > 1$ and let w have the maximal index among its conjugates. Put $k := \lfloor ind(w) \rfloor$ and denote w' the prefix of w such that

$$w^{ind(w)} = w^k w'.$$

Then

(i) *all the following factors are bispecial:*

$$w', ww', \ldots, w^{k-1}w',$$

(ii) *there exist $a, b \in \{0, 1\}$ so that $aw^k w'b$ is a factor of $\mathbf{u}_\beta$, $w'b$ is not a prefix of w and a is not the last letter of w.*

Bispecial factors in $\mathbf{u}_\beta$

Lemma

Let v be a bispecial factor of $\mathbf{u}_\beta$ containing at least once the letter 1. Then there exists unique bispecial factor $\tilde{v}$ such that

$$v = 0^{t_2} 1 \varphi_\beta(\tilde{v}) 0^{t_2} =: T(\tilde{v}),$$

i.e.,

each bispecial factor is either $0^s, s = 1, 2, \ldots, t_1 - 1$ or it is equal to the T-image of another bispecial factor or of the empty word.

Maximal powers in $\mathbf{u}_\beta$

Lemma

Each maximal power $w^{ind(w)}$ of a factor w having $ind(w) \geq t_1$ equals $T^k(0^{t_1})$ for a certain $k \in \{0, 1, 2, \ldots\}$.

Moreover,

$$\mathrm{E}(\mathbf{u}_\beta) = \sup\{ind(w^{(n)}) \mid n \in \mathbb{N}\},$$

where

$$w^{(1)} = 0, \qquad w^{(n+1)} = 0^{t_2}1\varphi_\beta(w^{(n)})(0^{t_2}1)^{-1}$$

and the maximal power

- *of $w^{(1)}$ is $v^{(0)} = 0^{t_1}$,*
- *of $w^{(n+1)}$ is $v^{(n+1)} = T(v^{(n)})$.*

Critical exponent of $\mathbf{u}_\beta$

The incidence matrix M is $\begin{pmatrix} |\varphi_\beta(0)|_0 & |\varphi_\beta(0)|_1 \\ |\varphi_\beta(1)|_0 & |\varphi_\beta(1)|_1 \end{pmatrix} = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \end{pmatrix}$.

For each factor u it holds $(|\varphi_\beta(u)|_0, |\varphi_\beta(u)|_1) = (|u|_0, |u|_1)M$,

Lemma

The number of 0s and 1s in the words $w^{(n)}$ and $v^{(n)}$ satisfy

$$(|w^{(n)}|_0, |w^{(n)}|_1) = (1,0)M^n,$$

$$(|v^{(n)}|_0, |v^{(n)}|_1) = (t_1+1, \tfrac{2t_2+1-t_1}{t_2})M^n - (1, \tfrac{2t_2+1-t_1}{t_2}).$$

$$\operatorname{ind}(w^{(n)}) = \frac{|v^{(n)}|}{|w^{(n)}|} =$$

$$= \frac{(t_1+1,0)M^n\begin{pmatrix}1\\1\end{pmatrix} + (0, \frac{2t_2+1-t_1}{t_2})M^n\begin{pmatrix}1\\1\end{pmatrix} - (1, \frac{2t_2+1-t_1}{t_2})\begin{pmatrix}1\\1\end{pmatrix}}{(1,0)M^n\begin{pmatrix}1\\1\end{pmatrix}}$$

Result

Theorem

If $t_1 \leq 3t_2 + 1$,

$$\mathrm{E}(\mathbf{u}_\beta) = t_1 + 1 + \frac{2t_2 + 1 - t_1}{\beta - 1},$$

otherwise

$$\mathrm{E}(\mathbf{u}_\beta) = ind(w^{(n_0)}) > t_1 + 1 + \frac{2t_2 + 1 - t_1}{\beta - 1}$$

for certain $n_0 \in \mathbb{N}$.

Result – general non-simple Parry case

Theorem

If β is a non-simple Parry number then we have

$$\mathrm{E}(\mathbf{u}_\beta) = \sup\{ind(w^{(n)}) \mid n \in \mathbb{N}\},$$

where

$$w^{(1)} = 0, \qquad w^{(n)} = \text{ conjugate of } \varphi_\beta^n(0)$$

and the maximal power

- *of $w^{(1)}$ is $v^{(0)} = 0^{t_1}$,*
- *of $w^{(n+1)}$ is $v^{(n+1)} = \tilde{T}(v^{(n)})$.*