

Fractional and integer colourings in claw-free graphs

Andrew King

KAM/ITI (Prague)
McGill University (Montreal)

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Joint work with Bruce Reed

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Our Motivation

We know $\chi \leq (1 + o(1))\chi_f$ for line graphs.

Can we extend this to broader classes of graphs?

Main Tool: Structure Theorem

$\alpha \geq 4 \rightarrow$ graph built as a generalization of a line graph.

We use restricted colourings of the “edges”.

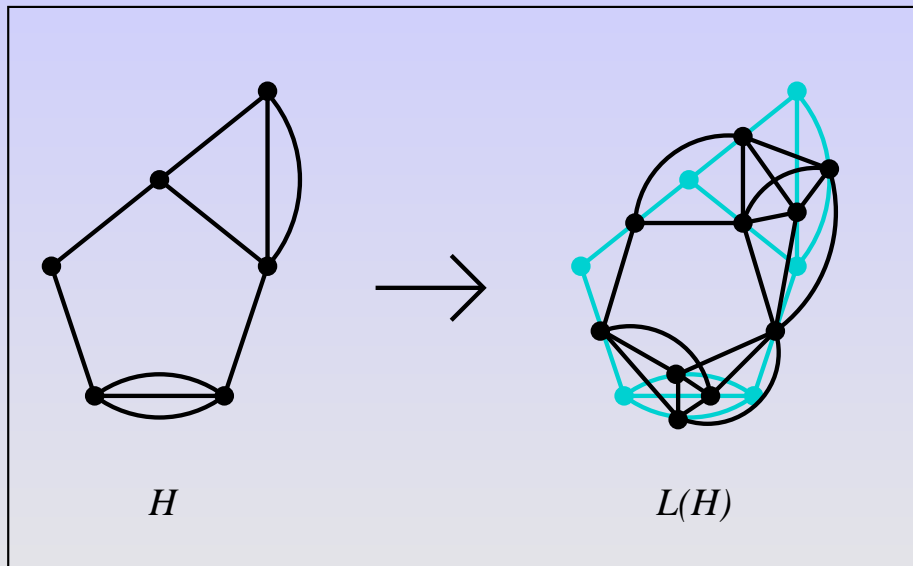
New Results

Quasi-line: $\chi \leq \chi_f + 3\sqrt{\chi_f}$.

Claw-free: $\chi \leq \chi_f + c\sqrt{\chi_f}$ for a constant c .

Bounds are achievable in polynomial time!

Line graph: $G = L(H)$



Colouring line graphs: $G = L(H)$

For a multigraph H , $\chi'(H) := \chi(L(H))$.

Goldberg-Seymour Conjecture (weakened)

For any line graph G , $\chi(G) \leq \chi_f(G) + 1$.

Theorem (Nishizeki, Kashiwagi '90)

For any H , $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H) + 0.8 \rfloor, \lceil \chi'_f(H) \rceil\}$.

Theorem (Kahn '96)

For any H , $\chi'(H) = (1 + o(1))\chi'_f(H)$.

Thus for line graphs, χ is close to χ_f .

Two natural generalizations

Definition

G is **quasi-line** if every vertex is **bisimplicial**.

(quasi-line = cobipartite neighbourhoods)

Definition

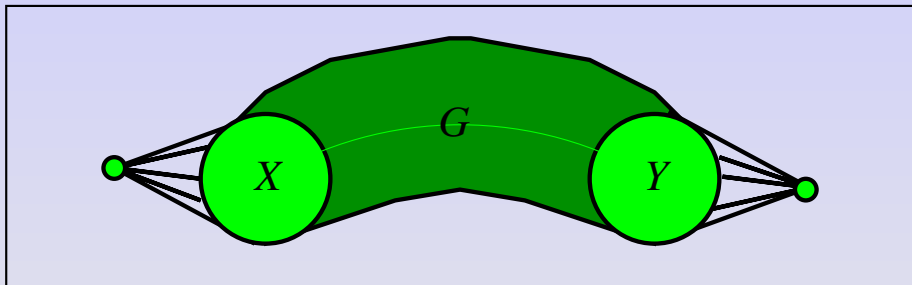
G is **claw-free** if it contains no induced $K_{1,3}$.

(claw-free = neighbourhoods have $\alpha \leq 2$)

line \subset quasi-line \subset claw-free

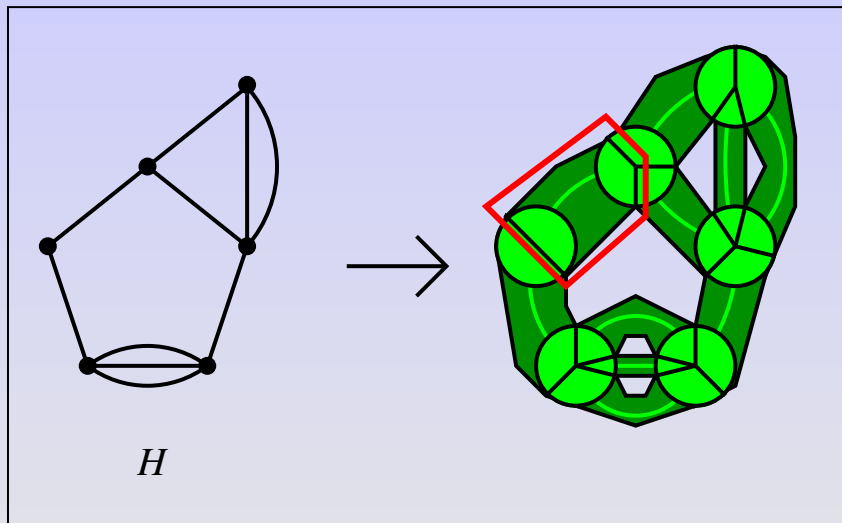
Strips

- A **strip** is like a claw-free graph with two “endpoints”.



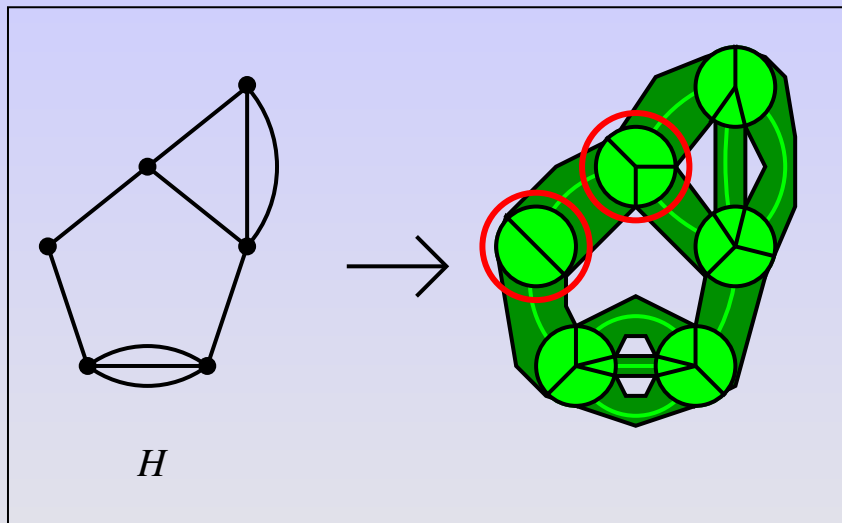
- Endpoints = “end-cliques” = neighbourhoods of two simplicial vertices, which we delete.

Compositions of strips



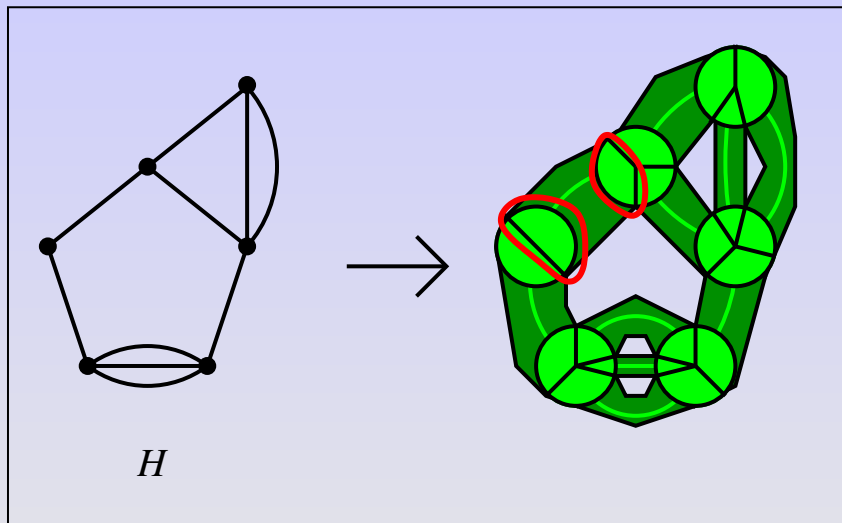
We replace each edge in H with a strip.

Compositions of strips



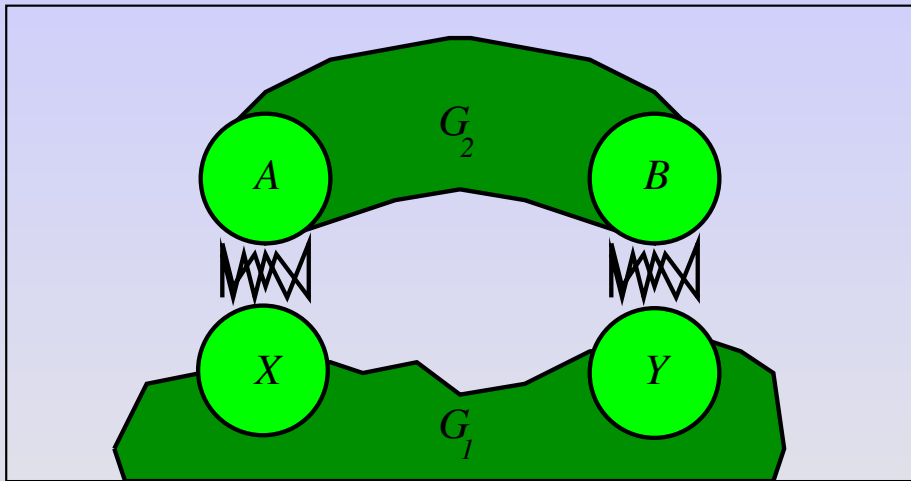
Vertices of H become **hub cliques**.

Compositions of strips



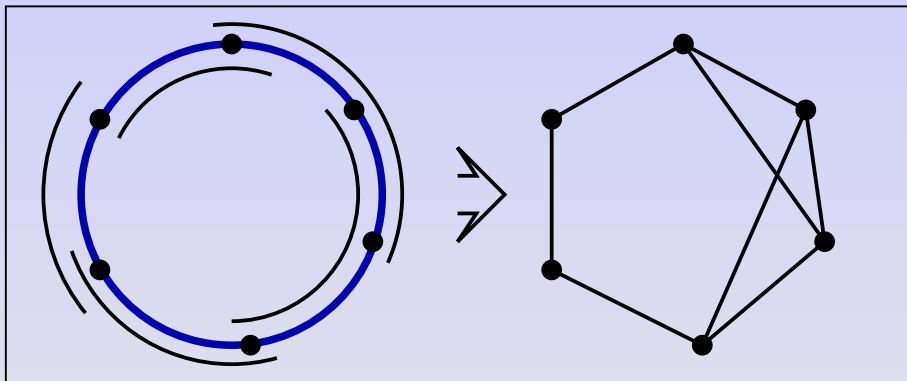
Each hub clique is made up of **end-cliques**.

Decomposition = 2-join



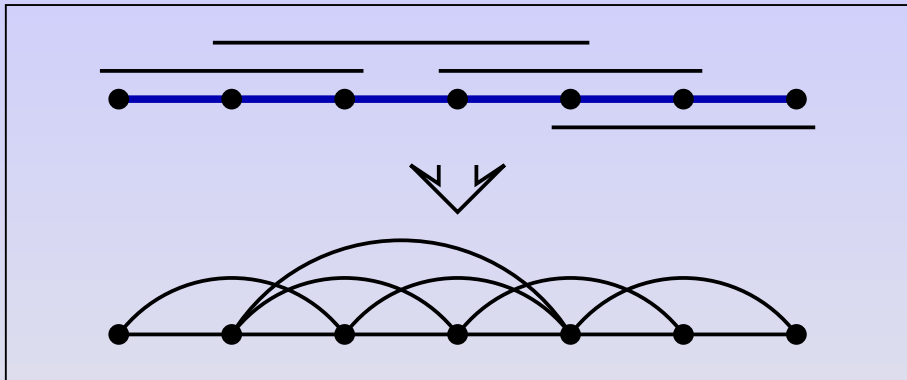
X and Y are the two end-cliques of a claw-free graph.

Circular Interval Graphs



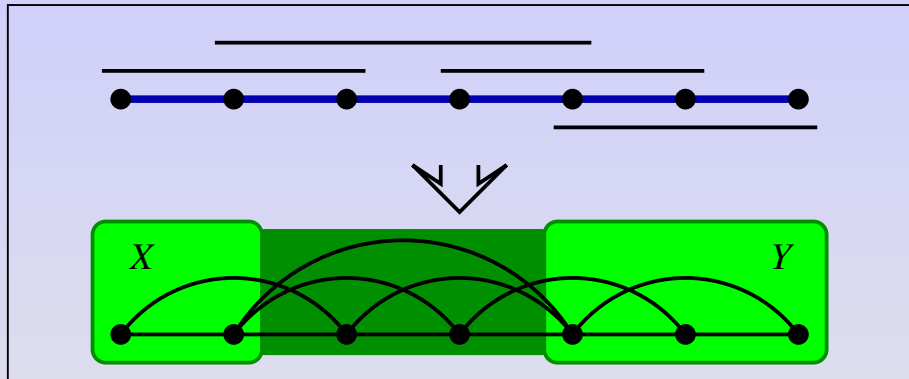
- Points on a **circle** \Rightarrow vertices.
- Intervals represent cliques.

Linear Interval Graphs



- Points on a **line** \Rightarrow vertices.
- Intervals represent cliques.

Linear Interval Strips



- End-cliques are at the left and right.
- End-cliques can overlap or even be equal.

The structure of quasi-line graphs

Theorem (Chudnovsky and Seymour)

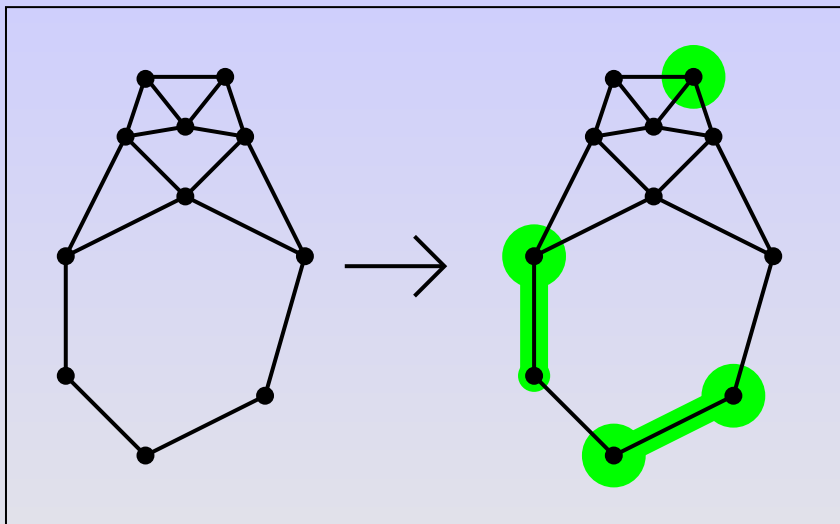
Let G be a quasi-line graph. One of the following holds:

- G is a circular interval graph.
- G is a composition of linear interval strips.
- G contains a homogeneous pair of cliques.

Now we need to deal with homogeneous pairs of cliques.

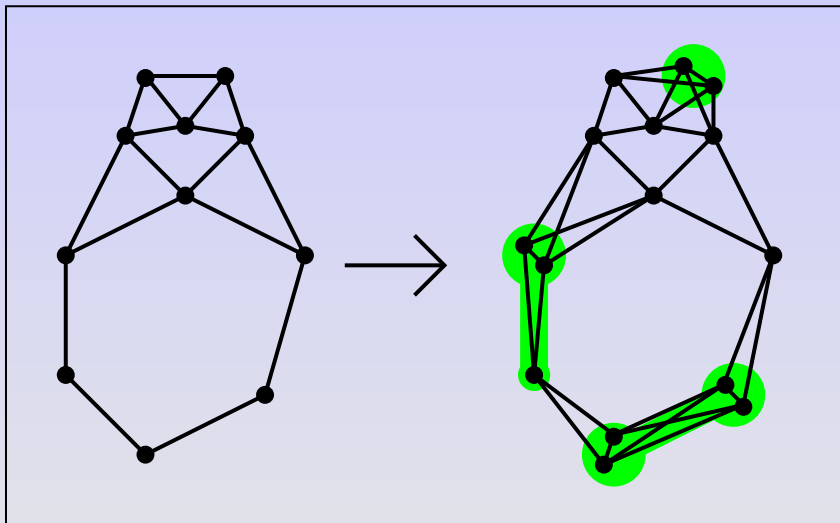
They arise mostly as a result of *thickenings* (vertex expansion)

Thickenings



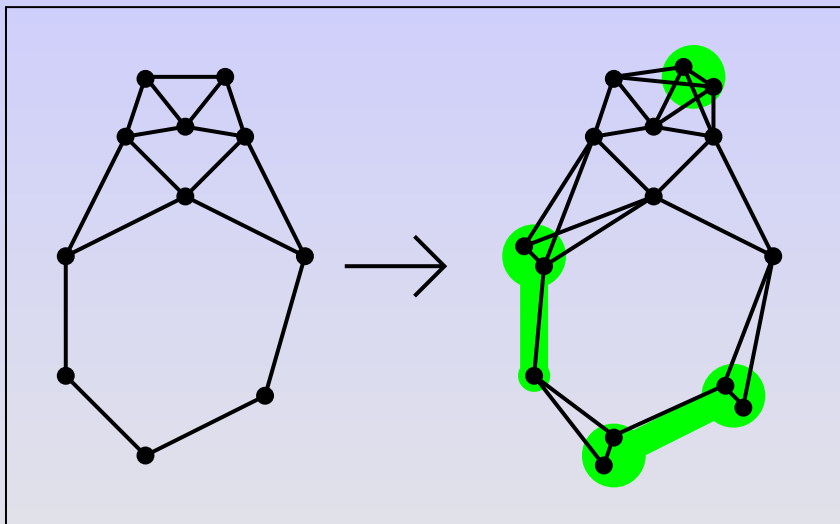
Choose a subset of vertices and a *claw-neutral* matching.

Thickenings



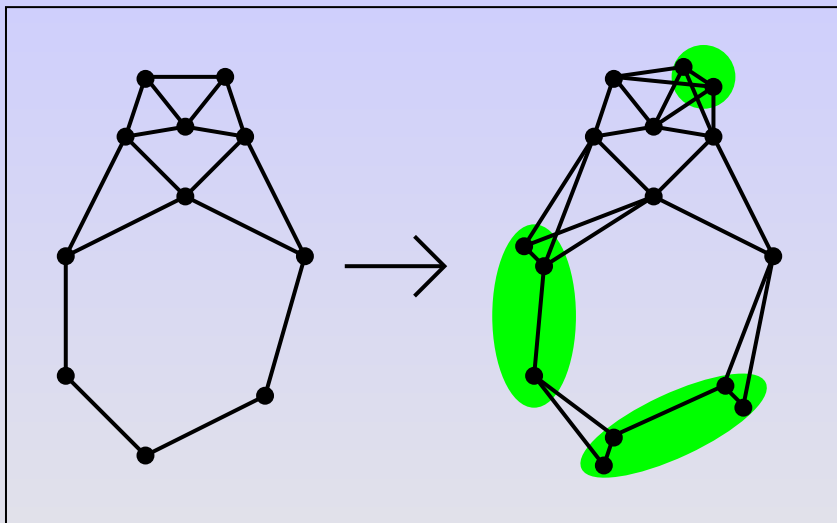
Blow up every vertex into a clique of *twins*.

Thickenings



Remove edges from special homogeneous pairs.

Thickenings



We get homogeneous cliques and homogeneous pairs of cliques.

Thickening

- Thickenings generalize *augmentations*.
- Quasi-line structure theorem generalizes *Berge quasi-line* structure theorem.

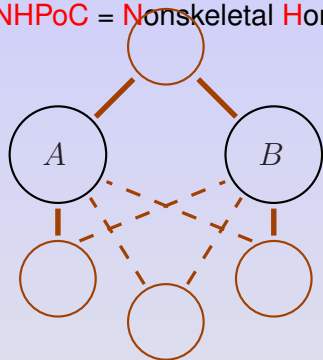
Theorem (Chudnovsky and Seymour)

Let G be a quasi-line graph. One of the following holds:

- G is a circular interval graph.
- G is a composition of linear interval strips.
- G contains a *nonskeletal* homogeneous pair of cliques.

Homogeneous pairs

NHPoC = **N**onskeletal **H**omogeneous **P**air of **C**liques (A, B):



Nonskeletal:

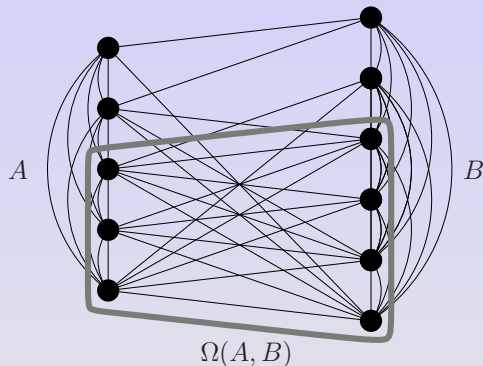
$G[A \cup B]$ has a maximum clique not containing all edges between A and B .

Lemma

If G contains an NHPoC, we can *remove edges without lowering $\chi(G)$ or $\chi_f(G)$* .

Reducing homogeneous pairs

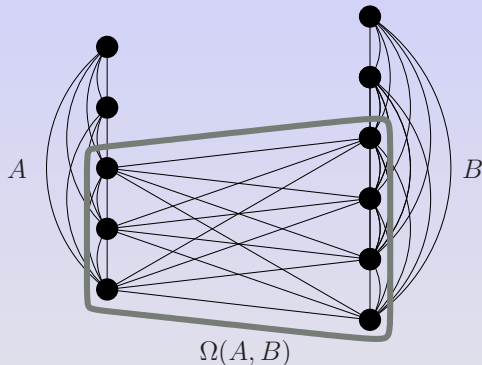
We can remove edges without changing ω , χ_f , or χ .



... until the structure is very simple!

Reducing homogeneous pairs

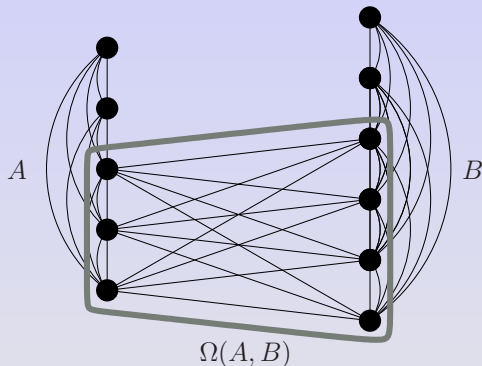
We can remove edges without changing ω , χ_f , or χ .



... until the structure is very simple!

Skeletal homogeneous pairs

The reduced homogeneous pair is **skeletal**.



A **skeletal** graph contains no nonskeletal homogeneous pair.

The structure of skeletal quasi-line graphs

Theorem

Let G be a skeletal quasi-line graph. One of the following holds:

- *G is a circular interval graph.*
- *G is a composition of linear interval strips.*

Key Idea

Because of our reductions, we need only consider skeletal graphs.

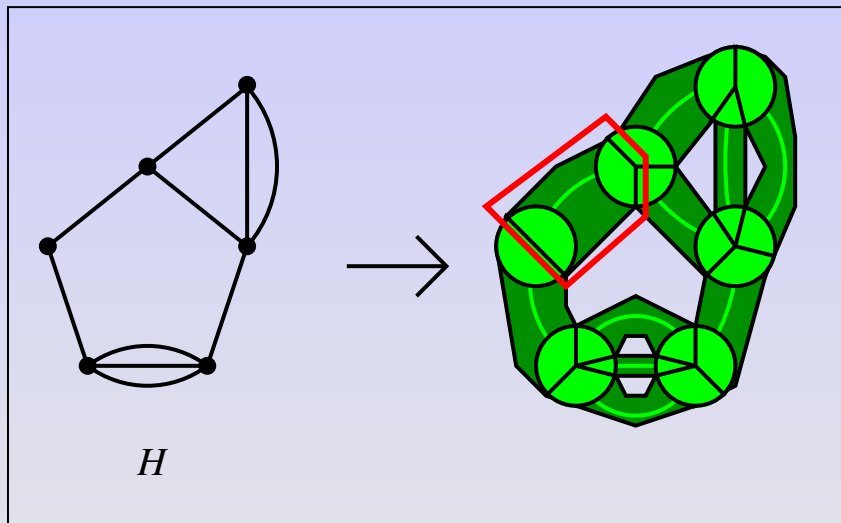
We get circular interval graphs for free.

Theorem (Niessen, Kind '00)

For any circular interval graph, $\chi = \lceil \chi_f \rceil$.

We also use this result to colour many auxiliary graphs.

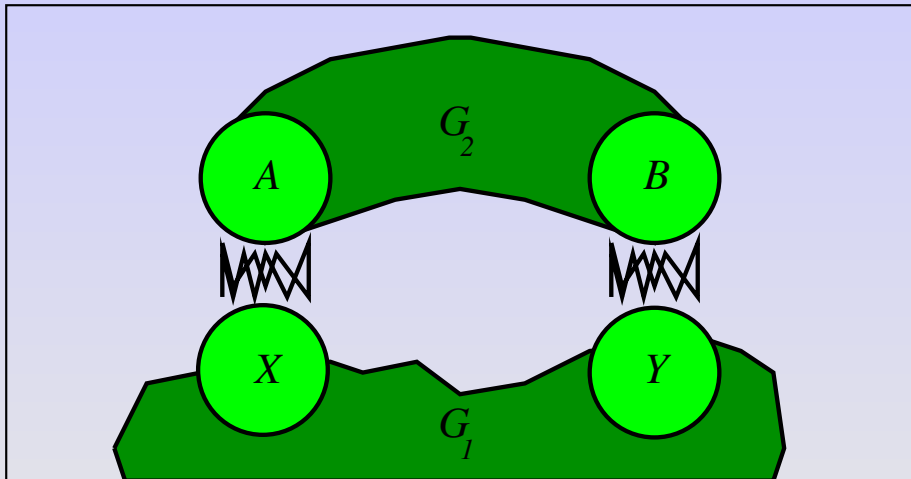
Compositions of strips



We **contract the strips** to find a colouring of the end-cliques.

Contracting with a fractional colouring

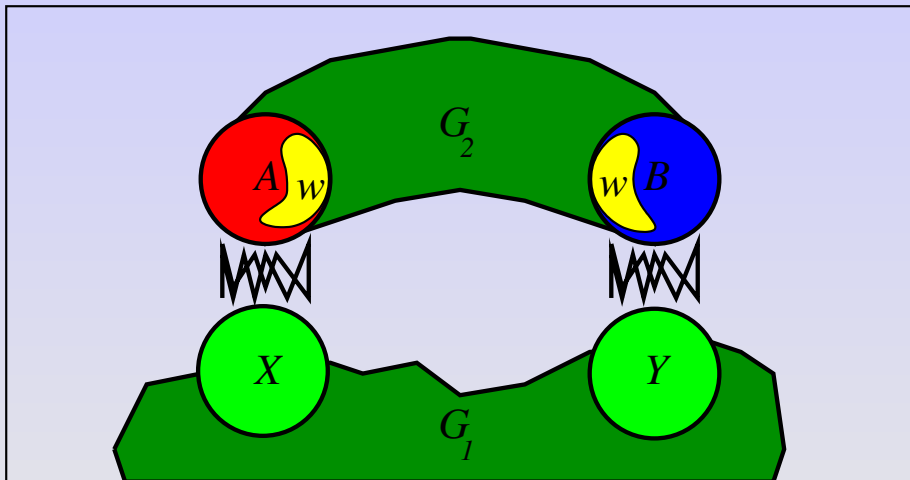
Fractionally colour the graph



Replace G_2 with a clique based on the colouring.

Contracting with a fractional colouring

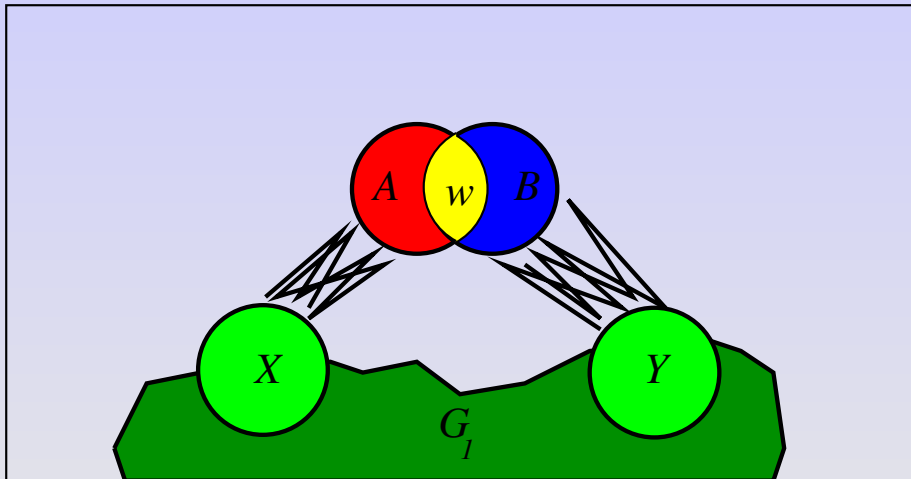
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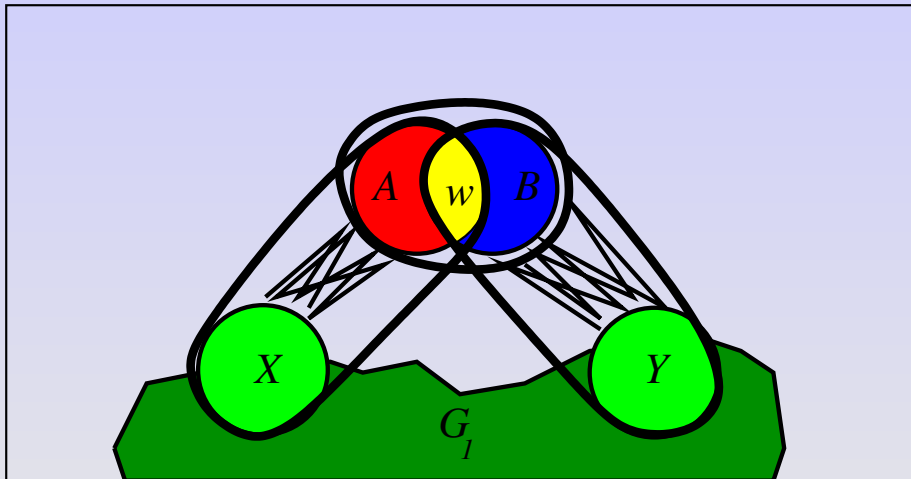
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Replace G_2 with a clique based on the colouring.

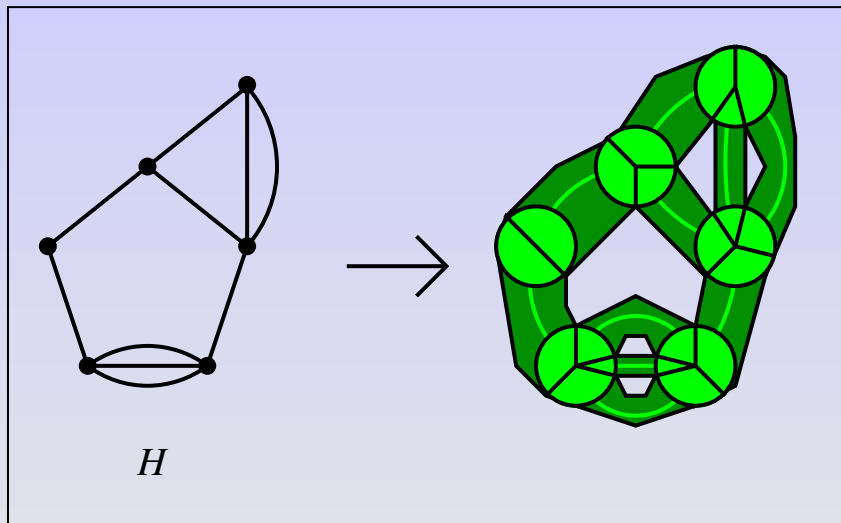
Contracting with a fractional colouring

Fractionally colour the graph



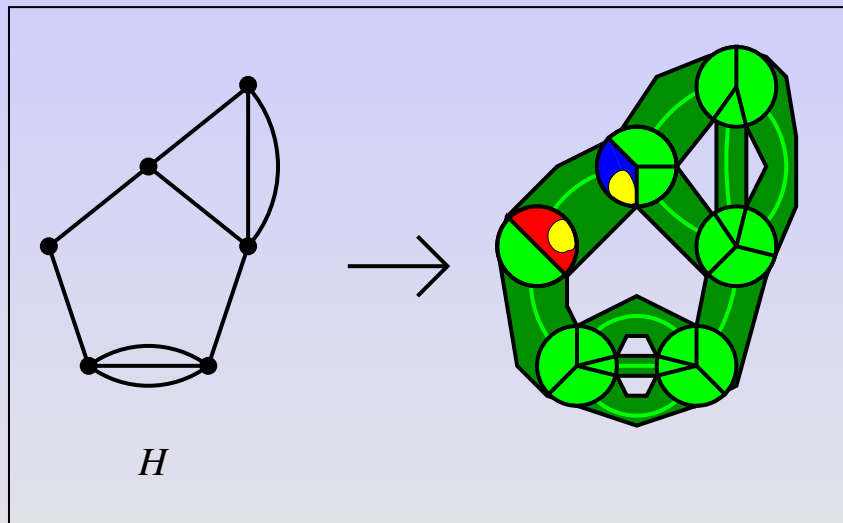
Replace G_2 with a clique based on the colouring.

Create a line graph from G



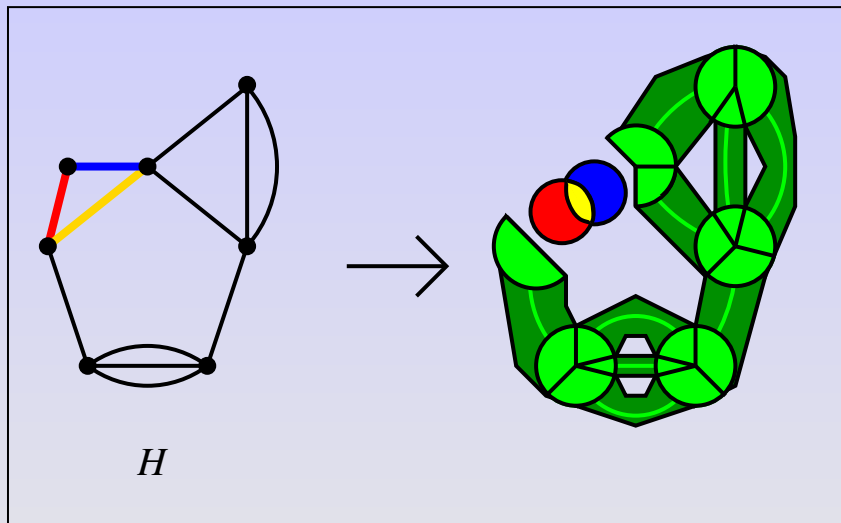
The contractions result in a line graph.

Create a line graph from G



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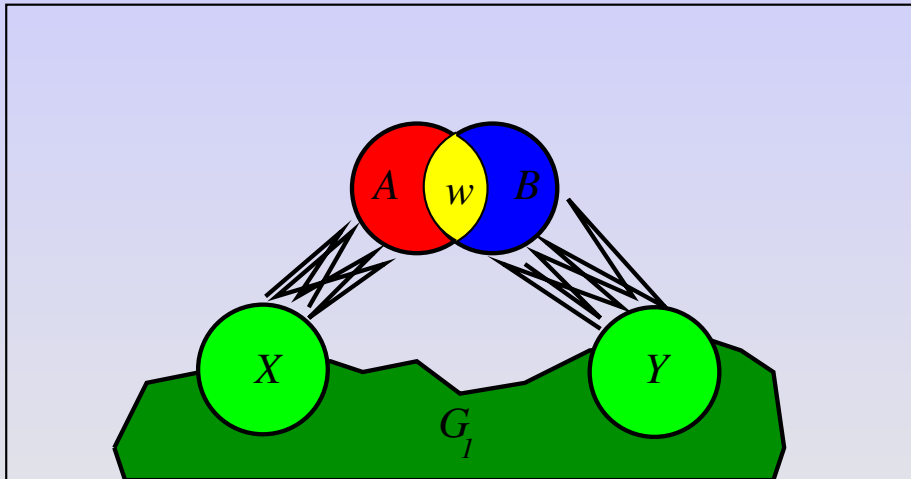
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The contractions result in a line graph.

Partially colour G

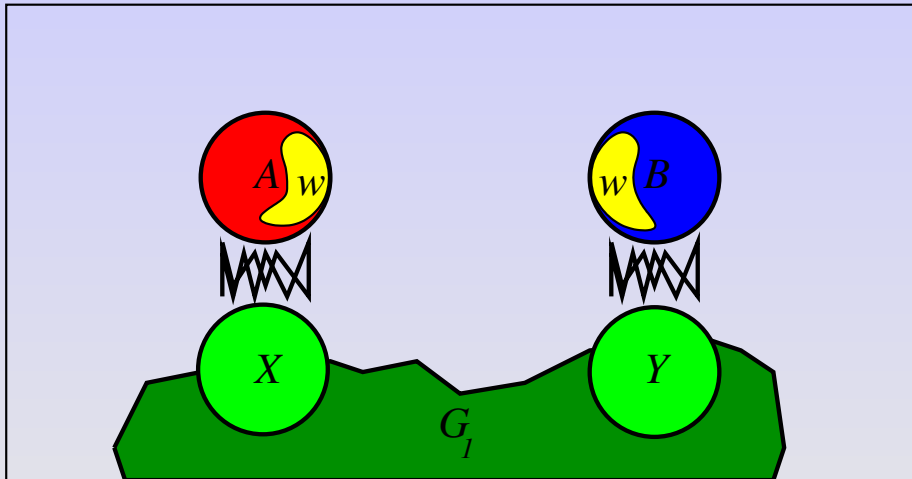
We get a colouring of the ends of the strips.



Now we fill in the rest.

Partially colour G

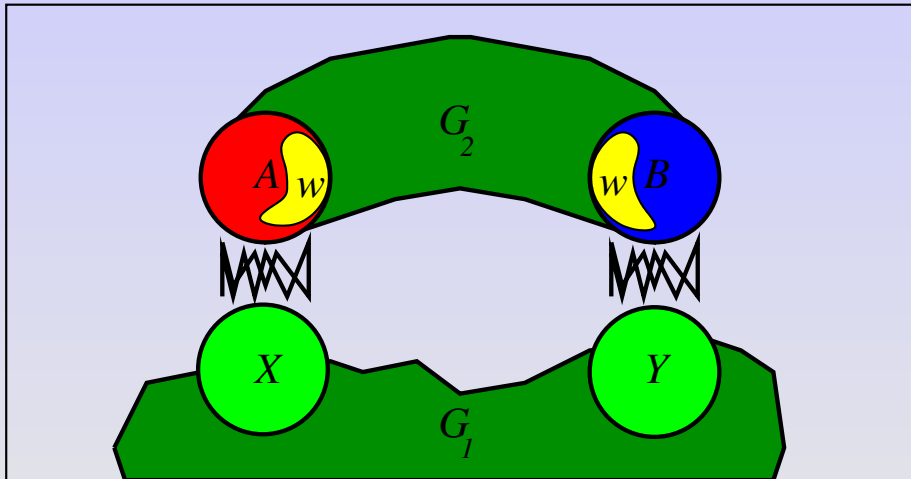
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Partially colour G

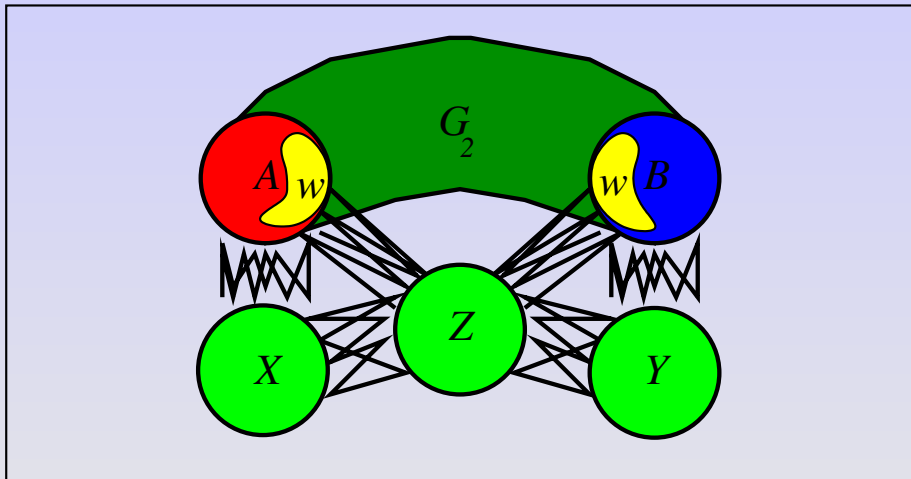
We get a colouring of the ends of the strips.



Now we fill in the rest.

Emulating fractional colourings

We need the right overlap each strip.



We colour an auxiliary circular interval graph.

Extending a theorem

Theorem (Sanders, Steurer)

For any *line* graph G , $\chi(G) \leq \chi_f(G) + \sqrt{\frac{9}{2}\chi_f(G)}$.

Theorem (K., Reed)

For any *quasi-line* graph G , $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$.

These bounds are achievable in polynomial time.

Example

Let G be k copies of C_5 joined together. Then $\chi(G) = 3k$. $\chi_f = \frac{5}{2}k$.

Conjecture

For any claw-free graph G , $\chi(G) \leq \lceil \frac{6}{5}\chi_f(G) \rceil$.

But what if we insist $\alpha(G) \geq 4$? (G must be connected)

Theorem (K., Reed)

For any claw-free graph G with $\alpha(G) \geq 4$,

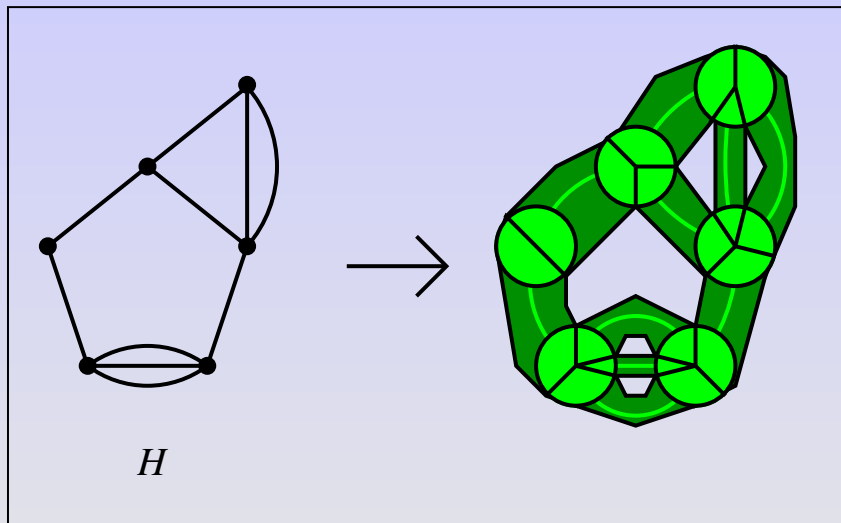
$$\chi(G) \leq \chi_f(G) + c\sqrt{\chi_f(G)}$$

Theorem (Chudnovsky, Seymour)

For G , a skeletal claw-free graph with $\alpha \geq 4$,

- G is a composition of strips (5 types), or
- G is a circular interval graph, or
- G admits a 1-join, one side of which is “simple”.

Why do we care about 1-joins?



1-joins arise from pendant edges in H .

Further applications of the structure theorem

More results on claw-free graphs:

- 1 (K., Reed) $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.
- 2 (Oriolo, Pietropaoli, Stauffer) New algorithm for MWSS problem.
- 3 (Chudnovsky, Fradkin) $\frac{3}{2}$ -approximation of Hadwiger's Conjecture.

And others.