A tribute to the memory of Michael O. Albertson, 1946 - 2009, and to his 40 years of graph theory research

Mike's theorems, conjectures and questions, and points of view that have influenced modern-day graph theory
• Thursday afternoon, "On graphs with crossings", organized by Mike and chaired by Debra Boutin
• March 26 - 28, 2010, a conference in honor Mike and his mathematics at Smith College
• Obituaries in AWM Newsletter, May - June, 2009, and (soon) in SIAM DM-Net.
• See www.math.smith.edu/faculty and click on Michael Albertson
Mike, David Berman, and I had the good fortune to study with Prof. Herbert Wilf at the University of Pennsylvania, late 60s, early 70s.

The Four Color Problem!

Erdős-Vizing Conjecture: Does every planar graph contain an independent set of at least $n/4$ vertices?

**Theorem** [A, 1976] Every planar graph contains an independent set of at least $2n/9$ vertices.

[A], [AH], [AB] … ? A = Albertson
Point of view:

For what proportion of the vertices of a graph can something be shown?

Conjecture [A, D, Berman, 1979] Every planar graph contains an induced forest with at least $n/2$ vertices.

A graph has an *acyclic coloring* with $k$ colors if each pair of color classes induces an acyclic subgraph.

AB (1977) proved every planar graph has an *acyclic 7-coloring*;

Borodin (1979) proved Grünbaum's conjecture (1973) that every planar graph has an acyclic 5-coloring, implying an induced forest with at least $2n/5$ vertices.
The length of the shortest noncontractible cycle in an embedded graph is called its width (1977).

**Theorem** [AHu, 1978], Every triangulation of a surface has a noncontractible cycle of length at most \( \sqrt{2n} \). Thus on a surface of genus \( g > 0 \) there is a planarizing set of size \( O(g\sqrt{n}) \).

Mike's favorite (Mike) conjecture (1980): All vertices of a triangulation of the torus can be 4-colored except for at most three vertices. aka Albertson's Four-Color Problem; see *Graph Coloring Prob-*
Theorems by Jensen & Toft.

Conjecture: For every surface $S$, there is an integer $f(S)$ such that all but $f(S)$ vertices of a graph embeddable on $S$ can be 4-colored.

Point of view:

Can you prove that all but $x$ vertices of a graph can be $k$-colored (for some $k$), and how small can $x$ be?
A&W Stromquist (1982): a *locally planar graph* is one embedded with all noncontractible cycles long and they proved that a *locally planar graph* on the torus could be 5-colored.
Generalizations of vertex-coloring - graph homomorphisms (preserves edges)

In 1985 with K. Collins, work on homomorphisms of 3-chromatic graphs, where graph automorphisms arose:
Let \( u(t, G) = \max\{\# \text{ vertices of } G \text{ in a } t\text{-colorable subgraph} \} \)

Consequence of the No Homomorphism Lemma: if \( f \) is a homomorphism from \( G \) to \( H \) and \( H \) is vertex-transitive, then

\[ u(t, G) \geq u(t, H) \text{ for each } t. \]

Mike also wrote additional papers on homomorphisms with several undergraduate students: L. Gibbons, V. Booth, L. Chan, and with Smith colleague Ruth Haas.
Also - circular chromatic number (D. West), vertex- and face-coloring of an embedded graph (B. Mohar), edge-coloring (RHaas, s-m belcastro), and list-coloring.

Ruth's favorite *conjecture* from "Partial List Coloring" (2000):

{Recall that if $G$ is $r$-colorable, then $u(t, G) \geq (t/r)n$.}

**Conjecture** [AHa, 2000]: Suppose $G$ is $s$-list-colorable, and suppose each vertex of $G$ is given a $t$-list for some $t < s$. Then at least $(t/s)n$ vertices can be $t$-list-colored.

**Open**: If $G$ is planar, can at least $n/2$ vertices be 2-list-colored?
Pre-coloring extensions, with A. Kostochka, E. Moore, D. West, and myself.

My candidate for another entry to Aigner and Ziegler's *Proofs from the Book*. 

*Proofs from the Book*
Carsten Thomassen asked in a preprint of "Color-critical graphs on a fixed surface":  

Suppose $G$ is a planar graph and $W \subseteq V(G)$ is such that the distance between every pair of vertices of $W$ is at least 100.  

Can every 5-coloring of $W$ be extended to a 5-coloring of $G$?  

[A, 1998] "You can't paint yourself into a corner" answered,  

"Yes, even with distance = 100, and both the parameters 5 and 4 are best possible."  

Carsten subsituted a conjecture about 2-coloring, far-apart bipartite subgraphs, and we proved that these always extend to a 5-coloring of the whole graph.  "You can't paint yourself into a corner" also has some nice open questions.
In 1996 Mike and Karen Collins began working on "symmetry breaking" in graphs, and introduced the distinguishing number $D(G)$:

A labeling $\phi: V(G) \to \{1, 2, \ldots, r\}$ is $r$-distinguishing if no nontrivial automorphism of $G$ preserves all vertex labels.

$$D(G) = \min\{r: G \text{ has a labeling that is } r\text{-distinguishing}\}.$$ 

This comes from a puzzle in "recreational math" which asks about

$$D(C_n) = 1, 2, 3, 3, 3, 3, \ldots$$
for $n = 1, 2, 3, 4, 5, \ldots$

$$D(C_n) = 2 \text{ for } n \geq 6.$$ 

[ABo, 2007] showed that for the Kneser graphs $K_{n:k}$, $n \geq 2k+1$, (two $k$-subsets of $[n]$ adjacent iff they intersect),

$$D(K_{n:1}) = 1, \ D(P = K_{5:2}) = 3, \text{ and}$$

$$D(K_{n:k}) = 2 \text{ otherwise.}$$
Mike's monikers:
  graphcolormike -- planargraphmike

[ABo, 2006] Distinguishing **Geometric Graphs**: graphs drawn in the plane with vertices in general position $\bar{G}$, with geometric automorphisms, automorphisms that preserve crossing and non-crossing edges,
  $$D(\bar{G}) \leq D(G), \text{ and}$$
  $$D_g(G) = \max \{D(\bar{G}) : \bar{G} \text{ is a geometric realization of } G\}$$
is the **geometric distinguishing number** of $G$ so that $D(\bar{G}) \leq D_g(G) \leq D(G)$.
**Theorem.** For $n = 3, 4, 5, 6$

$D_g(K_n) = 3, 4, 3, 3$.

**Question:** For $n \geq 7$, is $D_g(K_n) = 2$?

or is there $N$ such that for $n \geq N$,

$D_g(K_n) = 2$?

With similar results and questions on complete bipartite graphs.
This work naturally leads to that of crossing numbers:
[A, 2008] Chromatic number, independence ratio, and crossing number

Given a geometric graph, two crossings are called dependent if two edges of the crossings are incident with the same vertex, and otherwise they are independent.

**Conjecture:** If $G$ has a representation $\bar{G}$ with all crossings independent, then $\chi(G) \leq 5$.

Mike proved $\chi(G) \leq 6$.

Come to the Thursday afternoon session "On graphs with crossings", to Dan Král's talk...
One of Mike's adages:
in a talk:
    pick one idea to introduce, to illustrate, to develop, on which to prove results, to question, and to intrigue ...
• Mike's papers: math.smith.edu/faculty and click on Mike Albertson, and carbon.cudenver.edu/~egethner/MikeAlbertson.html

• If you'd like to be on the mailing list for the March 26 - 28 conference at Smith College, Northampton, MA, a conference in memory of Michael Albertson, please leave me or email me your name and email address, <hutchinson@macalester.edu>