Generating Self-Complementary Uniform Hypergraphs

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CanaDAM
May 25-28, 2008

Outline

Definitions

Generating self-complementary k-hypergraphs

Cycle types of antimorphisms

- Necessary and sufficient conditions on order
- A test
- An algorithm

k-hypergraphs and isomorphisms

- A **k-hypergraph (k-hypergraph)** X is a pair (V, E), in which V is a finite set of **vertices**, and E is a set of k-subsets of V called **edges**. The **rank** of X is k, and its **order** is $|V|$.
- An **isomorphism** between two k-hypergraphs $X = (V, E)$ and $Y = (W, F)$ is a bijection between V and W which induces a bijection between E and F. If such a bijection exists, X and Y are **isomorphic**.

Self-complementary k-hypergraphs

- The **complement** X^C of a k-hypergraph $X = (V, E)$ is the k-hypergraph
$$X^C = (V, E^C)$$
in which E^C is the set of k-subsets of V which are not in E.

- A k-hypergraph is **self-complementary (s.c.)** if it is isomorphic to its complement.

(Wojda, Szymański, 2007) For positive integers n and k, $k \leq n$, there exists a s.c. k-hypergraph of order n if and only if $\binom{n}{k}$ is even.

Antimorphisms, a.k.a. k-complementing permutations

- An **antimorphism** of a self-complementary k-hypergraph X is an isomorphism from X to its complement X^C.
- An antimorphism of a self-complementary k-hypergraph is sometimes called a **k-complementing permutation**.
- Odd powers of antimorphisms are antimorphisms.
- Antimorphisms have even order.
- Every self-complementary hypergraph has an antimorphism which has order a power of 2.

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An algorithm

Construct a s.c. 3-hypergraph of order 6.

$\theta = (1\ 2)(3\ 4\ 5\ 6) \in Sym(6)$

Let $A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$
$\{x, y, z\}$ $\{x, y, z\}^{\theta}$ $\{x, y, z\}^{\theta^2}$ $\{x, y, z\}^{\theta^3}$

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Construct a s.c. 4-hypergraph on $\mathbb{Z}_{18}$.

$\theta = (0\ 1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11)(12\ 13\ 14\ 15\ 16\ 17) \in Sym(\mathbb{Z}_{18})$

$A_1 = \{0, 3, 6, 9\}$ $\{1, 4, 7, 10\}$ $\{2, 5, 8, 11\}$

......

An orbit of θ on the 4-subsets of $\mathbb{Z}_{18}$ has odd length
$\implies$ θ is not a 4-complementing permutation in $Sym(\mathbb{Z}_{18})$.

A permutation $\theta \in Sym(V)$ is a k-complementing permutation

$\iff$ the sequence $A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$ has even length,

for all k-subsets A of V,

$\iff$ $A^{\theta^j} \neq A$,

for all k-subsets A of V, for all odd integers j.

If θ is a k-complementing permutation in $Sym(V)$, and there are m orbits of θ on the k-subsets of V, then **Algorithm 1** finds the set of 2^m self-complementary k-hypergraphs on V with antimorphism θ, called the **θ-switching class** of s.c. k-hypergraphs on V, denoted by $\mathcal{H}_{\theta}$.

Cycle types of antimorphisms

The cycle types of the k-complementing permutations have been characterized.

$k = 2$ **Sachs, Ringel (1962), Suprunenko (1985)**
$k = 3$ **Suprunenko (1985), Kocay (1992)**
$k = 4$ **Szymański (2005)**
Any k **Wojda, (2007)**

(Wojda, 2007) Let k, m and n be positive integers, let V be a finite set, $|V| = n$, and let $\sigma \in Sym(V)$ with orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, O_m$. Let $2^{q_i}(2s_i + 1)$ denote the cardinality of the orbit $\mathcal{O}_i$, for $i = 1, 2, \ldots, m$. The permutation σ is a k-complementing permutation if and only if, for every $\ell \in \{1, 2, \ldots, k\}$ and for every decomposition

$$k = k_1 + k_2 + \cdots + k_\ell$$

of k, where $k_j = 2^{p_j}(2r_j + 1)$ for nonnegative integers p_j, r_j, and for every subsequence of orbits $\mathcal{O}_{i_1}, \mathcal{O}_{i_2}, \ldots, \mathcal{O}_{i_\ell}$ such that $k_j \leq |\mathcal{O}_{i_j}|$ for $j = 1, 2, \ldots, \ell$, there is a subscript $j_0 \in \{1, 2, \ldots, \ell\}$ such that $p_{j_0} < q_{i_{j_0}}$.

Notation: $n_{[m]}$

For integers n, m, the symbol $\mathbf{n_{[m]}}$ denotes the unique integer $i \in \{0, 1, \ldots, m-1\}$ such that $n \equiv i (\bmod\ m)$.

Cycle types of antimorphisms

Theorem A (G, 2008)

- V is a finite set, $k \leq |V|$.
- b is the binary representation of k.
- $\theta \in Sym(V)$ has order a power of 2.
- Given $\ell \in support(b)$,
  - A_ℓ: set of points of V contained in cycles of θ of length $< 2^\ell$
  - B_ℓ: set of points of V contained in cycles of θ of length $> 2^\ell$.

Then θ is a k-complementing permutation if and only if, for some $\ell \in support(b)$, $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$.

Sketch of proof

($\Rightarrow$)

- $L := \{\ell \in support(b) : \theta$ has no cycle of length $2^\ell\} \neq \emptyset$.
- For some $\ell \in L$, $|A_\ell| < k_{[2^{\ell+1}]} = \sum_{i=0}^{\ell} b_i 2^i$.
  (Proof by contradiction - if not, then θ has an invariant set of size k.)

($\Leftarrow$)

- If $V = A_\ell \cup B_\ell$ and $|A_\ell| < k_{[2^{\ell+1}]}$ for some $\ell \in support(b)$, then θ has no invariant set of size k
- For each odd integer j, θ^j has the same cycle type as θ, and hence θ^j also has no invariant set of size k.
- Thus $A^{\theta^j} \neq A$ for all odd j and all k-subsets A of V.
- Hence θ is k-complementing.

Necessary and sufficient conditions on order

Corollary

Let k, n be positive integers, $k \leq n$, and let b be the binary representation of k. There exists a s.c. k-hypergraph of order n if and only if

$$n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}, \text{ for some } \ell \in \text{support}(b).$$

Necessary and sufficient conditions on order

Corollary

Let ℓ and r be integers, $\ell \geq 1$, $r \geq 0$.

1. *If $k = \sum_{i=0}^{r} 2^{\ell+i}$, then there exists a s.c. k-hypergraph of order n if and only if $n_{[2^{\ell+r+1}]} < k$.*
2. *If $k = 2^{\ell} - 1$, then there exists a s.c. k-hypergraph of order n if and only if $n_{[2^{\ell}]} < k$.*

Necessary and sufficient conditions on order

k	**n**
$2 = 2^1$	$n \equiv 0, 1 \pmod 4$
$3 = 2^1 + 1$	$n \equiv 0, 1, 2 \pmod 4$
$4 = 2^2$	$n \equiv 0, 1, 2, 3 \pmod 8$
$5 = 2^2 + 1$	$n \equiv 0, 1, 2, 3, 4, 6 \pmod 8$
$6 = 2^1 + 2^2$	$n \equiv 0, 1, 2, 3, 4, 5 \pmod 8$
$7 = 2^0 + 2^1 + 2^2$	$n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 8$
$8 = 2^3$	$n \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{16}$
$9 = 2^3 + 1$	$n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14 \pmod{16}$

A test

Lemma

Let V be a finite set, and let s be an integer. A permutation $\theta \in Sym(V)$ is a k-complementing permutation if and only if θ^{2s+1} is a k-complementing permutation.

Corollary

Let k be a positive integer, let b be the binary representation of k, and let V be a finite set. A permutation $\sigma \in Sym(V)$ is a k-complementing permutation if and only if $|\sigma| = 2^i(2t+1)$ for some integers t, i such that $i \geq 1$ and $t \geq 0$, and $\theta = \sigma^{2t+1}$ satisfies the conditions of Theorem A for some $\ell \in support(b)$.

This yields a test to determine if a given permutation in $Sym(V)$ is k-complementing.

An algorithm

Lemma

Every self-complementary k-hypergraph has an antimorphism whose order is equal to a power of 2.

Lemma

If θ and σ are conjugate k-complementing permutations in Sym(V), then each hypergraph in the θ-switching class $\mathcal{H}_\theta$ is isomorphic to a hypergraph in the σ-switching class $\mathcal{H}_\sigma$.

This yields an algorithm for generating all self-complementary k-hypergraphs of order n, up to isomorphism.

Algorithm 2

Input: Positive integers n and k, $k \leq n$.
Output: The set $\mathcal{H}$ of s.c. k-hypergraphs on $\{1, 2, \ldots, n\}$.

- Let b be the binary representation of k.
- Set $\mathcal{L} = \{\ell \in support(b) : n_{[2^{\ell+1}]} < k_{[2^{\ell+1}]}\}$.
- For each $\ell \in \mathcal{L}$, construct a set $\mathcal{S}_\ell$ of representatives of the conjugacy classes of permutations in $Sym(n)$ with order a power of 2 which satisfy the conditions of Theorem A for ℓ.
- Set
$$\mathcal{S} = \bigcup_{\ell \in \mathcal{L}} \mathcal{S}_\ell.$$
- Set
$$\mathcal{H} = \bigcup_{\theta \in S} \mathcal{H}_\theta.$$

Example: s.c. 3-hypergraphs of order 6

$3 = 2^0 + 2^1$
$6 \equiv 0 \pmod{2}$ and $6 \equiv 2 \pmod{4}$.
$6_{[2^{0+1}]} < 3_{[2^{0+1}]}$ and $6_{[2^{[1+1]}]} < 3_{[2^{1+1}]}$.
$\mathcal{L} = \{0, 1\}$
$S_0 = C((1\ 2)(3\ 4)(5\ 6)) \cup C((1\ 2)(3\ 4\ 5\ 6))$
$S_1 = C((1)(2)(3\ 4\ 5\ 6))$.

Example: s.c. 3-hypergraphs of order 6

(1 2)(3 4)(5 6)		(1 2)(3 4 5 6)				(1)(2)(3 4 5 6)			
123	124	123	124	125	126	345	456	356	346
134	234	134	245	156	236	134	145	156	136
125	126	234	145	256	136	234	245	256	236
156	256	345	456	356	346	123	124	125	126
345	346	135	246			135	146		
356	456	146	235			235	246		
135	246								
235	146								
136	245								
145	146								

There are at most $2^9 + 2^5 + 2^5 = 512 + 32 + 32 = 576$ s.c. 3-hypergraphs of order 6, up to isomorphism.

Thank You!

Input: Finite set V, positive integer $k \leq |V|$, $\theta \in Sym(V)$.
Output: YES if θ is a k-complementing permutation, NO otherwise.

(1) If $|\theta|$ is odd, output NO and quit. Otherwise, go to (2).

(2) Write $|\theta| = 2^i(2t+1)$ for some positive integer i.
Let $\hat{\theta} = \theta^{2t+1}$, and let $p = (n_1, n_2, \ldots, n_r)$ be the cycle type of $\hat{\theta}$. Set

$$L_p := \{\ell \in support(b) : n_i \neq 2^\ell \text{ for all } i \in \{1, 2, \ldots, r\}\}.$$

If $L_p = \emptyset$, output NO and quit. Otherwise go to (3).

(3) Choose $\ell \in L_p$. Let $n_0 = 0$, and let s be the largest integer such that $n_i < 2^\ell$ for all $i \leq s$. If $\sum_{i=0}^{s} n_i < \sum_{i=0}^{\ell} b_i 2^i$, output YES and quit. Otherwise go to (4).

(4) Set $L_p := L_p \setminus \{\ell\}$. If $L_p = \emptyset$, output NO and quit. Otherwise return to (3).