Nikolaos Fountoulakis[1] Bruce Reed [2]

A generalized critical condition for the emergence of a giant component in random graphs with given degrees

[1]Max-Planck-Institut für Informatik, Germany [2]McGill University, Montréal, Canada

2nd Canadian Discrete and Algorithmic Mathematics Conference, 26 May 2009

The structure of random graphs

The $G_{n,p}$ model (Erdős-Rényi)

Consider K_n (the complete graph on n vertices), and
retain each one of its edges, independently, with probability p.

- One of the main questions is:

  Given $p = p(n)$, is there a component with at least ϵn vertices, with probability that tends to 1 as $n \to \infty$?
  Such a component is a called a *giant component*.

The structure of random graphs

The $G_{n,p}$ model (Erdős-Rényi)

Consider K_n (the complete graph on n vertices), and
retain each one of its edges, independently, with probability p.

- One of the main questions is:

  Given $p = p(n)$, is there a component with at least ϵn vertices, with probability that tends to 1 as $n \to \infty$?
  Such a component is a called a *giant component*.

The structure of random graphs

The $G_{n,p}$ model (Erdős-Rényi)

Consider K_n (the complete graph on n vertices), and
retain each one of its edges, independently, with probability p.

- One of the main questions is:

  Given $p = p(n)$, is there a component with at least ϵn vertices, with probability that tends to 1 as $n \to \infty$?

  Such a component is a called a *giant component*.

The structure of random graphs

The $G_{n,p}$ model (Erdős-Rényi)

Consider K_n (the complete graph on n vertices), and retain each one of its edges, independently, with probability p.

- One of the main questions is:

  Given $p = p(n)$, is there a component with at least ϵn vertices, with probability that tends to 1 as $n \to \infty$?
  Such a component is a called a *giant component*.

The phase transition

A classical result by Erdős and Rényi:

- If $p > \frac{1+c}{n}$, then with probability $\to 1$, as $n \to \infty$, $G_{n,p}$ has a (unique) giant component, whereas every other component has $O(\log n)$ vertices;
- if $p < \frac{1-c}{n}$, then all the components of $G_{n,p}$ have $O(\log n)$ vertices, with probability $1 - o(1)$.

The phase transition

A classical result by Erdős and Rényi:

- If $p > \frac{1+c}{n}$, then with probability $\to 1$, as $n \to \infty$, $G_{n,p}$ has a (unique) giant component, whereas every other component has $O(\log n)$ vertices;
- if $p < \frac{1-c}{n}$, then all the components of $G_{n,p}$ have $O(\log n)$ vertices, with probability $1 - o(1)$.

The phase transition

A classical result by Erdős and Rényi:

- If $p > \frac{1+c}{n}$, then with probability $\to 1$, as $n \to \infty$, $G_{n,p}$ has a (unique) giant component, whereas every other component has $O(\log n)$ vertices;
- if $p < \frac{1-c}{n}$, then all the components of $G_{n,p}$ have $O(\log n)$ vertices, with probability $1 - o(1)$.

Random graphs with given degrees

For any integer $n \geq 1$, we let $\mathbf{d}_n = (d_1, \ldots, d_n)$ be a vector of non-negative integers with

$$\sum_{i=1}^{n} d_i \text{ even.}$$

- This is a *degree sequence*, in the sense that on the set of vertices $\{1, \ldots, n\}$,

$$\text{vertex } i \text{ has degree } d_i.$$

- We consider the set of all *simple* graphs on the vertex-set $\{1, \ldots, n\}$ whose degree sequence is $\mathbf{d}_n$.
- We let $G(\mathbf{d}_n)$ be a random graph uniformly chosen among all *simple* graphs whose degree sequence is $\mathbf{d}_n$.

Random graphs with given degrees

For any integer $n \geq 1$, we let $\mathbf{d}_n = (d_1, \ldots, d_n)$ be a vector of non-negative integers with

$$\sum_{i=1}^{n} d_i \text{ even.}$$

- This is a *degree sequence*, in the sense that on the set of vertices $\{1, \ldots, n\}$,

$$\text{vertex } i \text{ has degree } d_i.$$

- We consider the set of all *simple* graphs on the vertex-set $\{1, \ldots, n\}$ whose degree sequence is $\mathbf{d}_n$.
- We let $G(\mathbf{d}_n)$ be a random graph uniformly chosen among all *simple* graphs whose degree sequence is $\mathbf{d}_n$.

Random graphs with given degrees

For any integer $n \geq 1$, we let $\mathbf{d}_n = (d_1, \ldots, d_n)$ be a vector of non-negative integers with

$$\sum_{i=1}^{n} d_i \text{ even.}$$

- This is a *degree sequence*, in the sense that on the set of vertices $\{1, \ldots, n\}$,

$$\text{vertex } i \text{ has degree } d_i.$$

- We consider the set of all *simple* graphs on the vertex-set $\{1, \ldots, n\}$ whose degree sequence is $\mathbf{d}_n$.
- We let $G(\mathbf{d}_n)$ be a random graph uniformly chosen among all *simple* graphs whose degree sequence is $\mathbf{d}_n$.

Random graphs with given degrees

For any integer $n \geq 1$, we let $\mathbf{d}_n = (d_1, \ldots, d_n)$ be a vector of non-negative integers with

$$\sum_{i=1}^{n} d_i \text{ even.}$$

- This is a *degree sequence*, in the sense that on the set of vertices $\{1, \ldots, n\}$,

  vertex i has degree d_i.
- We consider the set of all *simple* graphs on the vertex-set $\{1, \ldots, n\}$ whose degree sequence is $\mathbf{d}_n$.
- We let $G(\mathbf{d}_n)$ be a random graph uniformly chosen among all *simple* graphs whose degree sequence is $\mathbf{d}_n$.

Why do we study them

- It is a "natural" model (e.g. random 3-regular graphs);
- many real-world random networks have (almost) fixed degrees.

Why do we study them

- It is a “natural" model (e.g. random 3-regular graphs);
- many real-world random networks have (almost) fixed degrees.

Why do we study them

- It is a "natural" model (e.g. random 3-regular graphs);
- many real-world random networks have (almost) fixed degrees.

The Molloy-Reed Theorem

- Let $D_i(n)$ be the number of elements in $\mathbf{d}_n = (d_1, \ldots, d_n)$ that are equal to i.

$$D_i(n) = \sharp \text{ vertices that have degree } i \text{ in } \mathbf{d}_n.$$

- Assume that $\lambda_i := \lim_{n\to\infty} \frac{D_i(n)}{n}$ exists and

$$\sum_{i\geq 1} i\lambda_i < \infty.$$

- Let

$$Q := \sum_{i\geq 1} i(i-2)\lambda_i.$$

The Molloy-Reed Theorem

- Let $D_i(n)$ be the number of elements in $\mathbf{d}_n = (d_1, \ldots, d_n)$ that are equal to i.

$$D_i(n) = \sharp \text{ vertices that have degree } i \text{ in } \mathbf{d}_n.$$

- Assume that $\lambda_i := \lim_{n\to\infty} \frac{D_i(n)}{n}$ exists and

$$\sum_{i \geq 1} i\lambda_i < \infty.$$

- Let

$$\mathcal{Q} := \sum_{i \geq 1} i(i-2)\lambda_i.$$

The Molloy-Reed Theorem

- Let $D_i(n)$ be the number of elements in $\mathbf{d}_n = (d_1, \ldots, d_n)$ that are equal to i.

$$D_i(n) = \sharp \text{ vertices that have degree } i \text{ in } \mathbf{d}_n.$$

- Assume that $\lambda_i := \lim_{n\to\infty} \frac{D_i(n)}{n}$ exists and

$$\sum_{i\geq 1} i\lambda_i < \infty.$$

- Let

$$\mathcal{Q} := \sum_{i\geq 1} i(i-2)\lambda_i.$$

The Molloy-Reed Theorem

Let $\Delta(n) = \max_i\{d_i\}$ be the maximum degree of the degree sequence $\mathbf{d}_n$.
Assume that the sequence of degree sequences $\{\mathbf{d}_n\}$ is *well-behaved*.

- If $\Delta(n) \leq n^{1/4-\epsilon}$ and $\mathcal{Q} > 0$, then

  $G(\mathbf{d}_n)$ has a component of order *at least* ϵn, with probability $\to 1$, as $n \to \infty$.

- If $\Delta(n) \leq n^{1/8-\epsilon}$ and $\mathcal{Q} < 0$, then

  the largest component of $G(\mathbf{d}_n)$ has no more than $\Delta(n)^2 \log(n)$ vertices, with probability $\to 1$, as $n \to \infty$.

The Molloy-Reed Theorem

Let $\Delta(n) = \max_i\{d_i\}$ be the maximum degree of the degree sequence $\mathbf{d}_n$.
Assume that the sequence of degree sequences $\{\mathbf{d}_n\}$ is *well-behaved*.

- If $\Delta(n) \leq n^{1/4-\epsilon}$ and $\mathcal{Q} > 0$, then

  $G(\mathbf{d}_n)$ has a component of order *at least* ϵn, with probability $\to 1$, as $n \to \infty$.

- If $\Delta(n) \leq n^{1/8-\epsilon}$ and $\mathcal{Q} < 0$, then

  the largest component of $G(\mathbf{d}_n)$ has no more than $\Delta(n)^2 \log(n)$ vertices, with probability $\to 1$, as $n \to \infty$.

The Molloy-Reed Theorem

Let $\Delta(n) = \max_i\{d_i\}$ be the maximum degree of the degree sequence $\mathbf{d}_n$.
Assume that the sequence of degree sequences $\{\mathbf{d}_n\}$ is *well-behaved*.

- If $\Delta(n) \leq n^{1/4-\epsilon}$ and $\mathcal{Q} > 0$, then

  $G(\mathbf{d}_n)$ has a component of order *at least* ϵn, with probability $\to 1$, as $n \to \infty$.

- If $\Delta(n) \leq n^{1/8-\epsilon}$ and $\mathcal{Q} < 0$, then

  the largest component of $G(\mathbf{d}_n)$ has no more than $\Delta(n)^2 \log(n)$ vertices, with probability $\to 1$, as $n \to \infty$.

Well-behaved degree sequences I

The term *Well-behaved* means:

- the ratios $i(i-2)D_i(n)/n$ converge uniformly to $i(i-2)\lambda_i$, for all $i \geq 1$.
  For every $\epsilon > 0$, there exists n_0 such that for all $i \geq 1$ and all $n > n_0$

$$\left| \frac{i(i-2)D_i(n)}{n} - i(i-2)\lambda_i \right| < \epsilon;$$

Well-behaved degree sequences I

The term *Well-behaved* means:

- the ratios $i(i-2)D_i(n)/n$ converge uniformly to $i(i-2)\lambda_i$, for all $i \geq 1$.
  For every $\epsilon > 0$, there exists n_0 such that for all $i \geq 1$ and all $n > n_0$
  $$\left|\frac{i(i-2)D_i(n)}{n} - i(i-2)\lambda_i\right| < \epsilon;$$

Well-behaved degree sequences II

If

$$\mathcal{L} := \lim_{n\to\infty} \sum_{i\geq 1} i(i-2)D_i(n)/n,$$

then the sum approaches the limit uniformly.

- if $\mathcal{L} < \infty$, then for all $\epsilon > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\left| \sum_{i=1}^{i^*} i(i-2)D_i(n)/n - \mathcal{L} \right| < \epsilon;$$

- if $\mathcal{L} = \infty$, then for all $T > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\sum_{i=1}^{i^*} i(i-2)D_i(n)/n > T.$$

Well-behaved degree sequences II

If

$$\mathcal{L} := \lim_{n\to\infty} \sum_{i\geq 1} i(i-2)D_i(n)/n,$$

then the sum approaches the limit uniformly.

- if $\mathcal{L} < \infty$, then for all $\epsilon > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\left| \sum_{i=1}^{i^*} i(i-2)D_i(n)/n - \mathcal{L} \right| < \epsilon;$$

- if $\mathcal{L} = \infty$, then for all $T > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\sum_{i=1}^{i^*} i(i-2)D_i(n)/n > T.$$

Well-behaved degree sequences II

If

$$\mathcal{L} := \lim_{n\to\infty} \sum_{i\geq 1} i(i-2)D_i(n)/n,$$

then the sum approaches the limit uniformly.

- if $\mathcal{L} < \infty$, then for all $\epsilon > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\left| \sum_{i=1}^{i^*} i(i-2)D_i(n)/n - \mathcal{L} \right| < \epsilon;$$

- if $\mathcal{L} = \infty$, then for all $T > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\sum_{i=1}^{i^*} i(i-2)D_i(n)/n > T.$$

Well-behaved degree sequences II

If

$$\mathcal{L} := \lim_{n\to\infty} \sum_{i\geq 1} i(i-2)D_i(n)/n,$$

then the sum approaches the limit uniformly.

- if $\mathcal{L} < \infty$, then for all $\epsilon > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\left| \sum_{i=1}^{i^*} i(i-2)D_i(n)/n - \mathcal{L} \right| < \epsilon;$$

- if $\mathcal{L} = \infty$, then for all $T > 0$ there exist i^* and n_0 such that for all $n > n_0$

$$\sum_{i=1}^{i^*} i(i-2)D_i(n)/n > T.$$

e two conditions imply that $\mathcal{Q} = \mathcal{L}$.

Our assumptions

- Our work

We aim to

the removal of the conditions regarding uniformity.

Our assumptions: with $D = D(n) = \sum_{i \geq 1} iD_i(n)$,

- For every $i \geq 1$, $\lim_{n \to \infty} \frac{iD_i(n)}{D(n)} =: \delta_i \geq 0$;
- $\Delta(\mathbf{d}_n) \leq D^{1/2-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant;
- $\frac{\sum_{i=1}^{n} d_i^2}{D} = O\left(D^{1/2-\epsilon}\right)$.

Our assumptions

- Our work

We aim to

the removal of the conditions regarding uniformity.

Our assumptions: with $D = D(n) = \sum_{i \geq 1} iD_i(n)$,

- For every $i \geq 1$, $\lim_{n \to \infty} \frac{iD_i(n)}{D(n)} =: \delta_i \geq 0$;
- $\Delta(\mathbf{d}_n) \leq D^{1/2-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant;
- $\frac{\sum_{i=1}^{n} d_i^2}{D} = O\left(D^{1/2-\epsilon}\right)$.

Our assumptions

- Our work

We aim to

the removal of the conditions regarding uniformity.

Our assumptions: with $D = D(n) = \sum_{i \geq 1} iD_i(n)$,

- For every $i \geq 1$, $\lim_{n\to\infty} \frac{iD_i(n)}{D(n)} =: \delta_i \geq 0$;
- $\Delta(\mathbf{d}_n) \leq D^{1/2-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant;
- $\frac{\sum_{i=1}^{n} d_i^2}{D} = O\left(D^{1/2-\epsilon}\right)$.

Our assumptions

- Our work

We aim to

the removal of the conditions regarding uniformity.

Our assumptions: with $D = D(n) = \sum_{i \geq 1} iD_i(n)$,

- For every $i \geq 1$, $\lim_{n\to\infty} \frac{iD_i(n)}{D(n)} =: \delta_i \geq 0$;
- $\Delta(\mathbf{d}_n) \leq D^{1/2-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant;
- $\frac{\sum_{i=1}^{n} d_i^2}{D} = O\left(D^{1/2-\epsilon}\right)$.

Our assumptions

- Our work

We aim to

the removal of the conditions regarding uniformity.

Our assumptions: with $D = D(n) = \sum_{i \geq 1} iD_i(n)$,

- For every $i \geq 1$, $\lim_{n \to \infty} \frac{iD_i(n)}{D(n)} =: \delta_i \geq 0$;
- $\Delta(\mathbf{d}_n) \leq D^{1/2-\epsilon}$, where $\epsilon > 0$ is an arbitrary constant;
- $\frac{\sum_{i=1}^{n} d_i^2}{D} = O\left(D^{1/2-\epsilon}\right)$.

For $s \in [0,1]$, we set

$$F(s) := \sum_{i \geq 1} \delta_i s^{i-1}.$$

Let s_0 be the smallest fixed point of F in $[0,1]$.

Theorem [F. and Reed]

Under our assumptions, with probability $1 - o(1)$

- if $s_0 < 1$, then the largest component of $G(\mathbf{d}_n)$ has at least cn vertices for some $c > 0$;
- if $s_0 = 1$, then *for every* $c > 0$ every component of $G(\mathbf{d}_n)$ has at most cn vertices.

For $s \in [0,1]$, we set

$$F(s) := \sum_{i \geq 1} \delta_i s^{i-1}.$$

Let s_0 be the smallest fixed point of F in $[0,1]$.

Theorem [F. and Reed]

Under our assumptions, with probability $1 - o(1)$

- if $s_0 < 1$, then the largest component of $G(\mathbf{d}_n)$ has at least cn vertices for some $c > 0$;
- if $s_0 = 1$, then *for every* $c > 0$ every component of $G(\mathbf{d}_n)$ has at most cn vertices.

For $s \in [0,1]$, we set

$$F(s) := \sum_{i \geq 1} \delta_i s^{i-1}.$$

Let s_0 be the smallest fixed point of F in $[0,1]$.

Theorem [F. and Reed]

Under our assumptions, with probability $1 - o(1)$

- if $s_0 < 1$, then the largest component of $G(\mathbf{d}_n)$ has at least cn vertices for some $c > 0$;
- if $s_0 = 1$, then *for every* $c > 0$ every component of $G(\mathbf{d}_n)$ has at most cn vertices.

Example

If we consider a degree sequence with $\lim_{n\to\infty} \frac{D_1(n)}{D}, \frac{2D_2(n)}{D} > 0$,

$$\frac{D_1(n)}{D} + \frac{2D_2(n)}{D} = 1 - o(1) \text{ and } n^{2/3} \text{ vertices of degree } n^{1/4},$$

our theorem implies that $G(\mathbf{d}_n)$ has not a giant component a.s..

However,

$$\sum_{i\geq 1} \frac{i(i-2)D_i(n)}{n} =: \mathcal{L} = \Theta\left(\frac{\left(n^{1/4}\right)^2 \cdot n^{2/3}}{n}\right) = \Theta\left(n^{1/6}\right)$$

but

$$\mathcal{Q} := \sum_{i\geq 1} i(i-2)\lambda_i = (-1)\lambda_1 < 0.$$

Example

If we consider a degree sequence with $\lim_{n\to\infty} \frac{D_1(n)}{D}, \frac{2D_2(n)}{D} > 0$,

$$\frac{D_1(n)}{D} + \frac{2D_2(n)}{D} = 1 - o(1) \text{ and } n^{2/3} \text{ vertices of degree } n^{1/4},$$

our theorem implies that $G(\mathbf{d}_n)$ has not a giant component a.s..

However,

$$\sum_{i\geq 1} \frac{i(i-2)D_i(n)}{n} =: \mathcal{L} = \Theta\left(\frac{\left(n^{1/4}\right)^2 \cdot n^{2/3}}{n}\right) = \Theta\left(n^{1/6}\right)$$

but

$$\mathcal{Q} := \sum_{i\geq 1} i(i-2)\lambda_i = (-1)\lambda_1 < 0.$$

Example

If we consider a degree sequence with

$$\frac{D_1(n)}{n} = 1 - o(1) \text{ and } n^{3/4} \text{ vertices of degree } n^{1/4},$$

our theorem implies that $G(\mathbf{d}_n)$ does have a giant component a.s..

But again in this case

$$\mathcal{L} = \sum_{i \geq 1} \frac{i(i-2)D_i(n)}{n} = \Theta\left(\frac{\left(n^{1/4}\right)^2 \cdot n^{3/4}}{n}\right) = \Theta\left(n^{1/4}\right)$$

and

$$\mathcal{Q} = (-1)1 < 0.$$

Example

If we consider a degree sequence with

$$\frac{D_1(n)}{n} = 1 - o(1) \text{ and } n^{3/4} \text{ vertices of degree } n^{1/4},$$

our theorem implies that $G(\mathbf{d}_n)$ does have a giant component a.s..

But again in this case

$$\mathcal{L} = \sum_{i \geq 1} \frac{i(i-2)D_i(n)}{n} = \Theta\left(\frac{\left(n^{1/4}\right)^2 \cdot n^{3/4}}{n}\right) = \Theta\left(n^{1/4}\right)$$

and

$$\mathcal{Q} = (-1)1 < 0.$$

Proof Idea I

We study the *core* of $G(\mathbf{d}_n)$.

The *core* of a graph G the maximum subgraph of minimum degree at least two.

The core of G can be recovered by repeatedly removing the vertices of degree 1 of G.

We refer to this process as the *stripping process*.

Proof Idea I

We study the *core* of $G(\mathbf{d}_n)$.

The *core* of a graph G the maximum subgraph of minimum degree at least two.

The core of G can be recovered by repeatedly removing the vertices of degree 1 of G.

We refer to this process as the *stripping process*.

Proof Idea I

We study the *core* of $G(\mathbf{d}_n)$.

The *core* of a graph G the maximum subgraph of minimum degree at least two.

The core of G can be recovered by repeatedly removing the vertices of degree 1 of G.

We refer to this process as the *stripping process*.

Proof Idea I

We study the *core* of $G(\mathbf{d}_n)$.

The *core* of a graph G the maximum subgraph of minimum degree at least two.

The core of G can be recovered by repeatedly removing the vertices of degree 1 of G.

We refer to this process as the *stripping process*.

Proof Idea II

- We have proved the following:

Theorem [F. and Reed]

Under our assumptions, the core of $G(\mathbf{d}_n)$ has total degree

$$D(1 - s_0)^2 + o(D).$$

- So

Summary

The giant component emerges, when a core of linear total degree emerges.

- This had been observed for degree sequences satisfying stronger conditions by Fernholz and Ramachandran (2003).

Proof Idea II

- We have proved the following:

Theorem [F. and Reed]

Under our assumptions, the core of $G(\mathbf{d}_n)$ has total degree

$$D(1 - s_0)^2 + o(D).$$

- So

Summary

The giant component emerges, when a core of linear total degree emerges.

- This had been observed for degree sequences satisfying stronger conditions by Fernholz and Ramachandran (2003).

Proof Idea II

- We have proved the following:

Theorem [F. and Reed]

Under our assumptions, the core of $G(\mathbf{d}_n)$ has total degree

$$D(1 - s_0)^2 + o(D).$$

- So

Summary

The giant component emerges, when a core of linear total degree emerges.

- This had been observed for degree sequences satisfying stronger conditions by Fernholz and Ramachandran (2003).

Proof Idea II

- We have proved the following:

Theorem [F. and Reed]

Under our assumptions, the core of $G(\mathbf{d}_n)$ has total degree

$$D(1 - s_0)^2 + o(D).$$

- So

Summary

The giant component emerges, when a core of linear total degree emerges.

- This had been observed for degree sequences satisfying stronger conditions by Fernholz and Ramachandran (2003).

Proof Idea II

- We have proved the following:

Theorem [F. and Reed]

Under our assumptions, the core of $G(\mathbf{d}_n)$ has total degree

$$D(1 - s_0)^2 + o(D).$$

- So

Summary

The giant component emerges, when a core of linear total degree emerges.

- This had been observed for degree sequences satisfying stronger conditions by Fernholz and Ramachandran (2003).

Proof Idea III

- $s_0 < 1$

  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that

  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then

     it contains a component of total degree at least, say, ϵD and a linear number of vertices of degree 1 attach to it.

Proof Idea III

- $s_0 < 1$
  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that
  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then

     it contains a component of total degree at least, say, ϵD and a linear number of vertices of degree 1 attach to it.

Proof Idea III

- $s_0 < 1$
  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that
  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then
     it contains a component of total degree at least, say, ϵD
     and a linear number of vertices of degree 1 attach to it.

Proof Idea III

- $s_0 < 1$
  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that
  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then

     it contains a component of total degree at least, say, ϵD
     and a linear number of vertices of degree 1 attach to it.

Proof Idea III

- $s_0 < 1$
  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that
  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then

     it contains a component of total degree at least, say, ϵD
     and a linear number of vertices of degree 1 attach to it.

Proof Idea III

- $s_0 < 1$
  We get a core of total degree $\approx D(1 - s_0)^2$ and we show that
  1. if the core contains a linear number of vertices, then it has a component with linear number of vertices;
  2. if the core does not contain a linear number of vertices, then
     it contains a component of total degree at least, say, ϵD
     and a linear number of vertices of degree 1 attach to it.

Proof Idea IV

- $s_0 = 1$

  We expose the components that contain the vertices of high degree (at least $\log^2 n$)

  we show that their total degree does not exceed ϵD with high probability.

  Then we condition on these components, and we show that this is the case for the remaining vertices.

Proof Idea IV

- $s_0 = 1$

  We expose the components that contain the vertices of high degree (at least $\log^2 n$)

  we show that their total degree does not exceed ϵD with high probability.

  Then we condition on these components, and we show that this is the case for the remaining vertices.

Proof Idea IV

- $s_0 = 1$

  We expose the components that contain the vertices of high degree (at least $\log^2 n$)

  we show that their total degree does not exceed ϵD with high probability.

  Then we condition on these components, and we show that this is the case for the remaining vertices.

Proof Idea IV

- $s_0 = 1$

  We expose the components that contain the vertices of high degree (at least $\log^2 n$)

  we show that their total degree does not exceed ϵD with high probability.

  Then we condition on these components, and we show that this is the case for the remaining vertices.

Some natural questions:

- determine if there is a unique giant component;
- determine its size;
- determine the size of the second largest component;
- determine the structure of the small components.

Some natural questions:

- determine if there is a unique giant component;
- determine its size;
- determine the size of the second largest component;
- determine the structure of the small components.

Some natural questions:

- determine if there is a unique giant component;
- determine its size;
- determine the size of the second largest component;
- determine the structure of the small components.

Some natural questions:

- determine if there is a unique giant component;
- determine its size;
- determine the size of the second largest component;
- determine the structure of the small components.

Some natural questions:

- determine if there is a unique giant component;
- determine its size;
- determine the size of the second largest component;
- determine the structure of the small components.

Thank you!