Enumeration of Fillings of Ferrers diagrams

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Canadam 09 - May 25th, 2009

Dedication

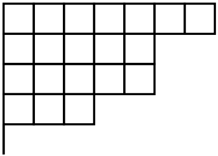
Pierre Leroux

Coworkers

- M. Josuat-Vergès (LRI - CNRS et Université Paris-Sud)
- J.S. Kim (LRI - CNRS et Université Paris-Sud)
- P. Nadeau (U. Wien)
- L.K. Williams (Harvard)

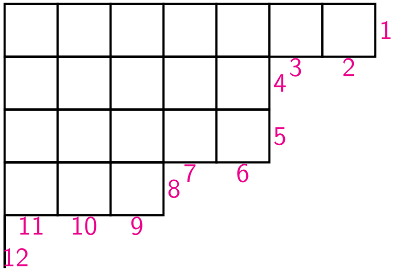
Ferrers diagram

$\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$
$\lambda = (7, 5, 5, 3, 0)$



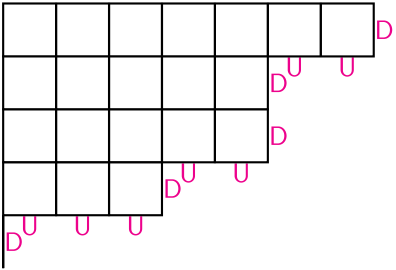
Length=number of rows + number of columns=$\lambda_1 + k$

Number



Number the border : 1,2,. . . ,12

Number



Code the border: *DUUDDUUDUUUD*

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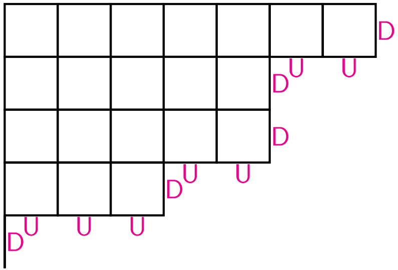

Code the border: *DUUDDUUDUUUD*

Number of Ferrers diagrams of length n : 2^{n-1}

Example I

$\lambda = (n, n, \ldots, n)$
One 1 in each row and column : Permutation matrix
The number of PM of length $2n$ is $n!$.

An inversion in a permutation is a couple (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$.

Question: What is the generating function

$$\sum_{\sigma \in \mathfrak{S}_n} q^{inv(\sigma)}?$$

$\sigma = 412635$

			1		
					1
1					
				1	
		1			
	1				

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			1	×	×
					1
1	×	×		×	
				1	
		1			
	1				

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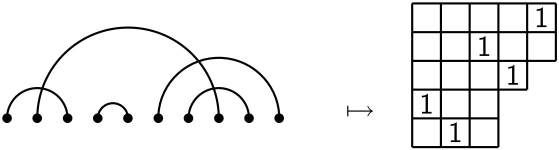
$$\sum_{\sigma \in \mathfrak{S}_n} q^{inv(\sigma)} = \prod_{i=1}^{n} (1 + \ldots + q^{i-1}) = [n]_q!$$

Example I (continued)

Any λ and one 1 per row and per column : Rook placements

The number of RP of length $2n$ is $(2n-1)!!$.



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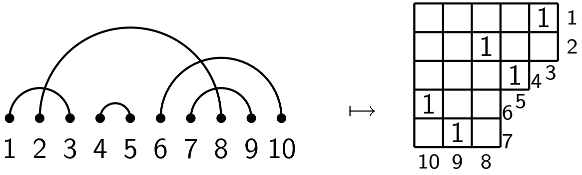

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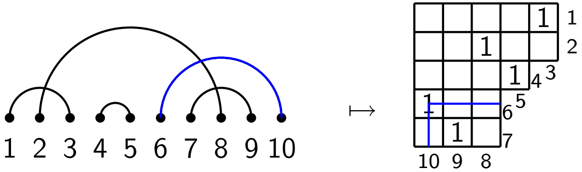

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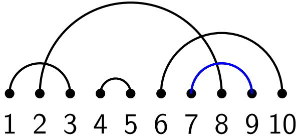
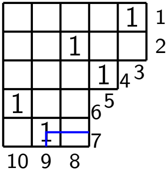


$\longmapsto$



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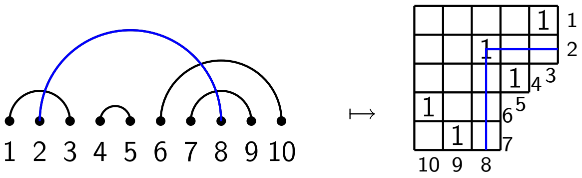

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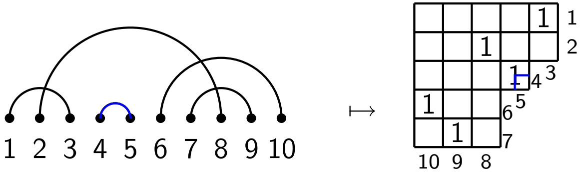

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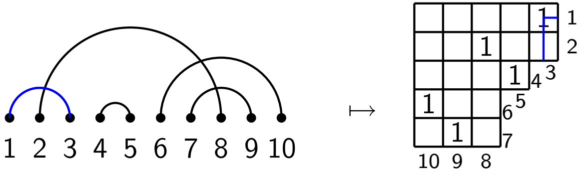

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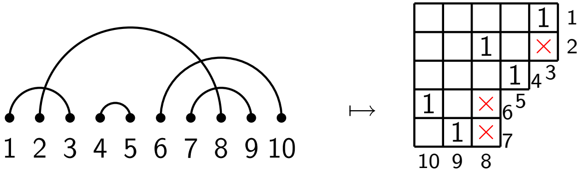
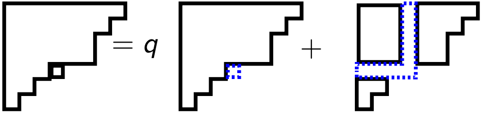


Crossings $\leftrightarrow$ Inversions

$$\sum_{m \in \mathfrak{M}_{2n}} q^{cr(m)} = \sum_{\text{rook placement}} q^{inv(R)}$$

Example I (cont.)



$$\sum_{\text{rook placement}} q^{inv(R)} = \langle W|(D+U)^{2n}|V\rangle$$

where $DU = qUD + I, \quad \langle W|U = 0, \quad D|V\rangle = 0, \quad \langle W||V\rangle = 1.$

$$\sum_{\text{rook placement}} q^{inv(R)} = \frac{1}{(1-q)^n}\sum_{i=0}^{n}(-1)^i\left(\binom{2n}{n-i}-\binom{2n}{n-i-1}\right)q^{\frac{i(i+1)}{2}}.$$

(Touchard 50s, Riordan 70s)

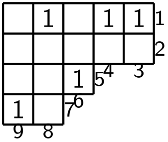
Example II

Pierre Leroux (88) 0-1 tableaux : One 1 per column.

The number of tableaux of length n is B_n (the n^{th} Bell number)

Set partitions $\mapsto$ 0-1 tableaux

$$\pi = (1,3,4,8)(2)(5,6)(7,9) \qquad \mapsto$$



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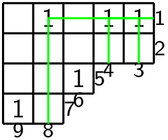

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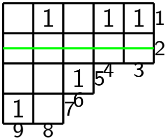

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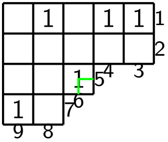

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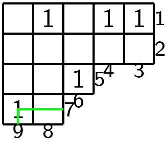

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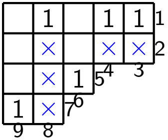

$$S_q(n,k) = \sum_{\substack{\text{0-1 tableaux} \\ \text{length } n,\ k \text{ rows}}} q^{inv(T)}$$

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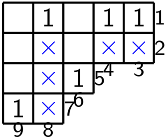

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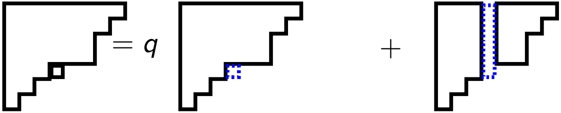
q-Log concavity (Leroux 88)

$$S_q(n,k)^2 - S_q(n,k-1)S_q(n,k+1) \geq_q 0.$$

Example II (cont.)



Enumeration

$$S_q(n,k) = [y^k]\langle W|(yD+U)^n|V\rangle$$

with $DU = qUD + U$, $\langle W|E = 0$, $D|V\rangle = 1$.

$$S_q(n,k) = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{n-k} (-1)^j \binom{n}{k+j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q$$

(Wachs and White 88)

Permutation tableaux (Postnikov 01, Williams 04)

Origin : Totally non negative part of the Grassmanian

Permutation tableau $\mathcal{T}$: a Ferrers diagram filled with 0's and 1's such that :

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

0	0	1	0	0	1	1
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1						

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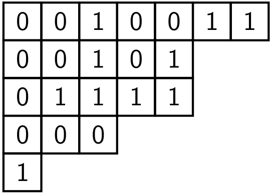

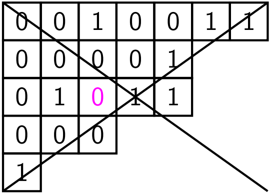

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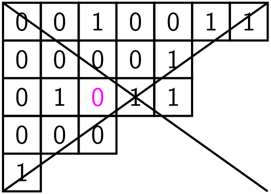
0	1	1	1	1		
0	0	0				
1						



Number of permutation tableaux of length n is $n!$

Permutation tableaux and alternative tableaux

Restricted zero : lies below some 1

A permutation tableau is uniquely defined by its topmost ones and rightmost restricted zeros.

(C. Nadeau 07)

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

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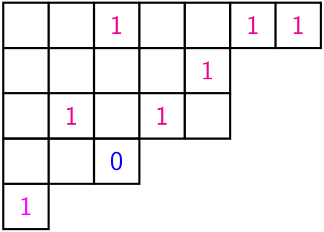

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0	0		0	1		
0	1		1			
0	0	0				
1						

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		1			1	1
		1		1		
	1	1	1	1		
		0				
1						

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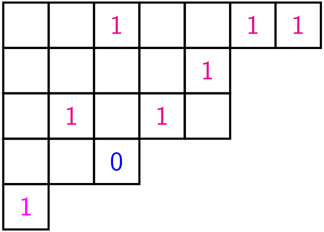
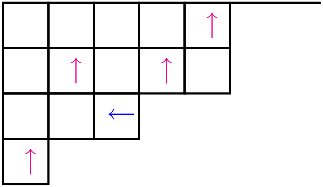


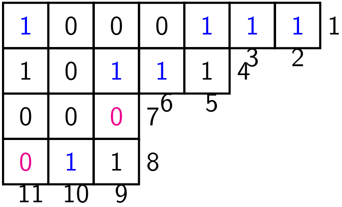


Alternative tableaux (Viennot 08, Nadeau 09)

Permutation tableaux and permutations

Columns ↔ Descents

(C. and Nadeau 07)



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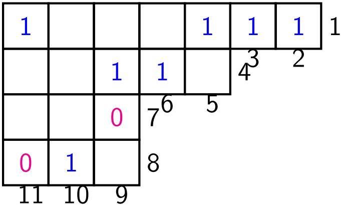

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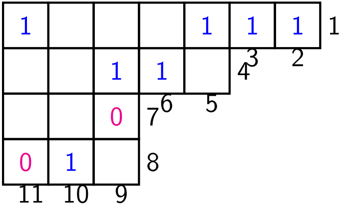
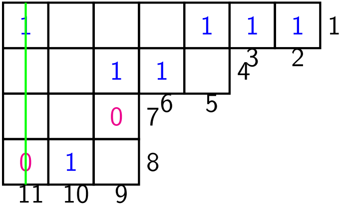


$(1,4)$

Permutation tableaux and permutations

Columns $\leftrightarrow$ Descents

(C. and Nadeau 07)



$(8,11,1,4)$

Permutation tableaux and permutations

Columns ↔ Descents

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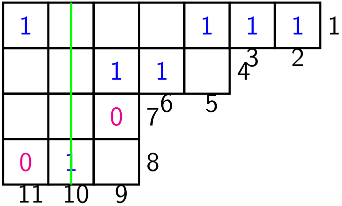
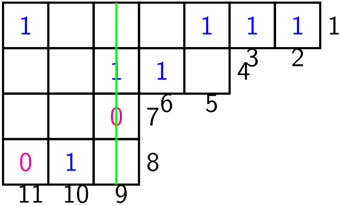


(10,8,11,1,4)

Permutation tableaux and permutations

Columns ↔ Descents

(C. and Nadeau 07)



$(10,8,11,1,7,9,4)$

Permutation tableaux and permutations

Columns ↔ Descents

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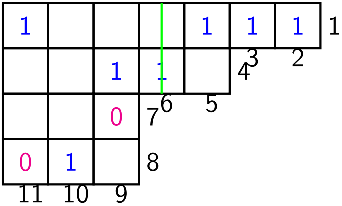

$$(10,8,11,1,7,9,6,4)$$

Permutation tableaux and permutations

Columns $\leftrightarrow$ Descents

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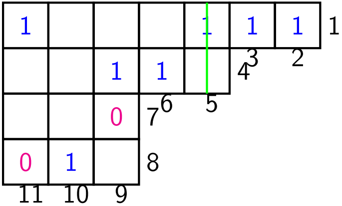


$$(10,8,11,5,3,2,1,7,9,6,4)$$

Other bijections

- Postnikov 01, Steingrimsson and Williams 05 : excedances and crossings.
- Burstein 05 : cycles
- C. and Nadeau 07 : descents and 31-2.
- Viennot 07 : descents

Enumeration of PT

- $u(\mathcal{T})$: number of unrestricted rows minus one
- $f(\mathcal{T})$: number of ones in the first row

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$f(\mathcal{T}) = 3$

$u(\mathcal{T}) = 4 - 1 = 3$

$$\sum_{\mathcal{T} \text{ length } n+1} x^{u(\mathcal{T})} y^{f(\mathcal{T})} = \prod_{i=0}^{n-1} (x + y + i) = (x + y)_n.$$

(C. and Nadeau 07)

q-enumeration of PT of a given shape

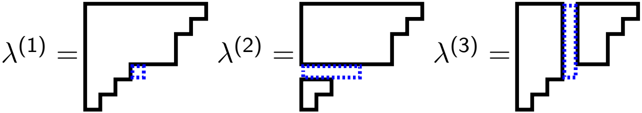
wt($\mathcal{T}$): number of ones minus number of columns

1	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		

$\mathrm{wt}(\mathcal{T}) = 10 - 7 = 3$

$F_\lambda(q) = \sum_{\mathcal{T} \text{ shape } \lambda} q^{\mathrm{wt}(\mathcal{T})}$

For any λ and a given corner, we define smaller Young diagrams:



$F_\lambda(q)$ is defined by the recurrence

$$F_\lambda = qF_{\lambda^{(1)}} + F_{\lambda^{(2)}} + F_{\lambda^{(3)}}; \qquad F_\emptyset = 1$$

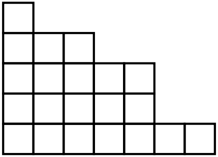
(Williams 05)

q-enumeration of PT of a given shape

As columns $\leftrightarrow$ descents

Non commutative symmetric functions (Tevlin 07)

$\lambda = (7, 5, 5, 3, 1)$



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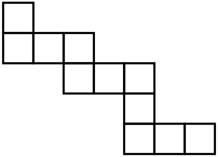


$I(\lambda) = (1, 3, 3, 1, 3)$

$$F_\lambda(q) = e_{I(\lambda)}(q) = \sum_{J \preceq I} (-q)^{\ell(J)-\ell(I)} q^{-\mathrm{st}'(I,J)} \prod_{k=1}^{p} [k]_q^{j_k}.$$

(Novelli, Thibon, Williams 08)

q-enumeration (cont.)

$$E_{k,n}(q) = \sum_{\substack{\lambda \\ \ell(\lambda)=k \\ \text{length } n}} F_\lambda(q)$$

q-enumeration of PT of length n with k rows

$$E_{k,n}(q) = q^{n-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right)$$

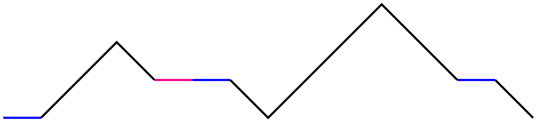
(Williams 05)

q-analogue of Eulerian numbers
$q = 0$ Narayana numbers, $q = -1$ Binomial numbers

Moments of q-Laguerre polynomials

(Kasraoui and Zeng 09)

PT permutation tableaux and Motzkin paths



$$F_n(q) = \sum_{\lambda \text{ length } n} F_\lambda(q) = \frac{1}{(1-q)^n} \sum_{p \text{ length } n} w(p)$$

Weight of each step starting at height h is

- East : $1 - q^{h+1}$ or $1 - q^h$
- North-East : $1 - q^{h+1}$
- South-East : $1 - q^h$

q-Laguerre Polynomials

$$F_n(q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \sum_{j=0}^{k} q^{j(k-j+1)}$$

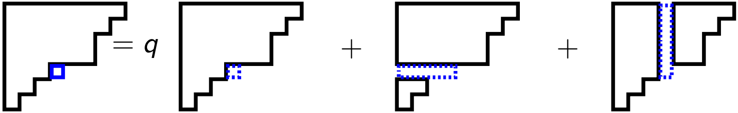
(C. Josuat-Vergès, Prellberg, Rubey 09)

PT permutation tableaux and Matrix Ansatz

$$F_{n+1}(q,\alpha,\beta) = \sum_{\mathcal{T} \text{ length } n+1} q^{\mathrm{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})},$$

$$F_{n+1}(q,\alpha,\beta) = \langle W|(D+U)^n|V\rangle, \qquad \text{where}$$

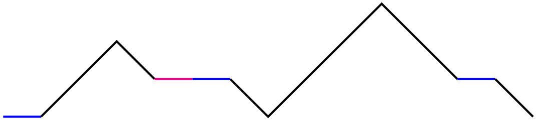
$$DU = qUD + D + U;$$



$$\alpha\langle W|E = \langle W|; \quad \beta D|V\rangle = |V\rangle \quad \langle W||V\rangle = 1.$$

(C. Williams 06)

PT permutation tableaux and Motzkin paths



$$F_{n+1}(q,\alpha,\beta) = \frac{1}{(1-q)^n} \sum_{p \text{ length } n} w(p)$$

$$\tilde{\alpha} = \frac{q-1}{\alpha} + 1; \quad \tilde{\beta} = \frac{q-1}{\beta} + 1$$

Weight of each step starting at height h is

- East : $1 - \tilde{\alpha} q^h$ or $1 - \tilde{\beta} q^h$
- North-East : $1 - q^{h+1}$
- South-East : $1 - \tilde{\alpha}\tilde{\beta} q^{h-1}$

(Brak, C. Essam, Parviainen,Rechnitzer 05)

Enumeration of PT (The end)

$$F_{n+1}(q,\alpha,\beta) = \frac{1}{(1-q)^n} \sum_{m=0}^{n} R_{n,m}(q) B_m(\tilde{\alpha},\tilde{\beta};q),$$

with

$$D_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-2}; \quad B_m(\tilde{\alpha},\tilde{\beta};q) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \tilde{\alpha}^{m-k}\tilde{\beta}^{k};$$

$$R_{n,m} = \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} (-1)^k D_{n,m+2k} q^{\binom{k+1}{2}} \begin{bmatrix} m+k \\ k \end{bmatrix}_q$$

(Josuat-Vergès 09)

PT and Partially asymmetric exclusion process

Model : n sites that are empty or occupied
The sites are delimited by $n+1$ positions ($n-1$ positions in between sites, left border and right border).

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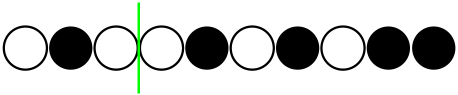

PT and Partially asymmetric exclusion process

Model : n sites that are empty or occupied
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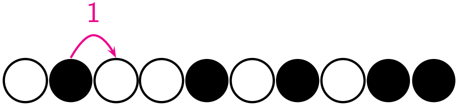

PT and Partially asymmetric exclusion process

Model : n sites that are empty or occupied
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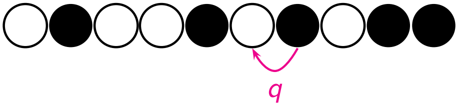
- First a position is chosen at random
- A particle hops to the right with probability 1
- A particle hops to the left with probability q



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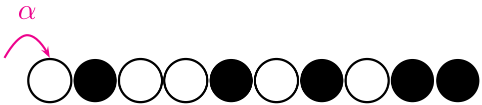

PT and Partially asymmetric exclusion process

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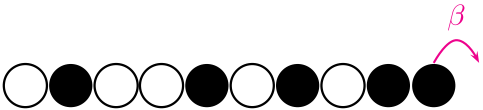
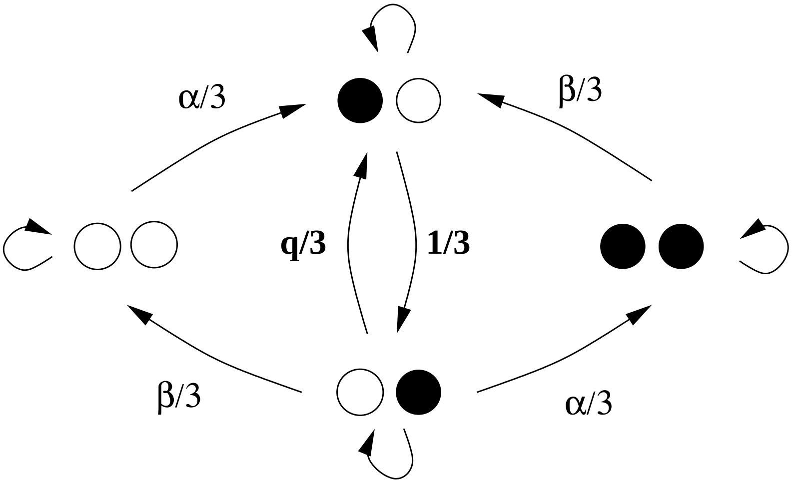
- A particle hops to the left with probability q
- A particle enters with probability α
- A particles leaves with probability β



Markov chain $n = 2$



Stationary distribution of the PASEP chain

$\leftrightarrow \qquad \tau = (0,0,1,0,0,1,1,0,0)$

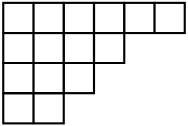
Let $P_n^{q,\alpha,\beta}(\tau)$ be the probability to be in state $\tau = (\tau_1, \ldots, \tau_n)$.
Theorem. (Derrida et al. 93) The probability to be in state $\tau = (\tau_1, \ldots, \tau_n)$ is

$$P_n(\tau) = \frac{\langle W | (\prod_{i=1}^{n} (\tau_i D + (1-\tau_i)E)) | V \rangle}{Z_n}.$$

with $Z_n = \langle W | (D+E)^n | V \rangle$, D and E are infinite matrices, V is a column vector, and W is a row vector, such that

$$DE - qED = D + E$$
$$\beta D | V \rangle = | V \rangle$$
$$\alpha \langle W | E = \langle W |$$

Permutation tableaux

$\tau = (0,0,1,0,0,1,1,0,0) \leftrightarrow \lambda(\tau) =$ 

Theorem. Fix $\tau = (\tau_1, \ldots, \tau_n) \in \{0,1\}^n$, and let $\lambda := \lambda(\tau)$. The probability of finding the PASEP chain in configuration τ in the steady state is

$$\frac{F_\lambda(q,\alpha,\beta)}{F_{n+1}(q,\alpha,\beta)}.$$

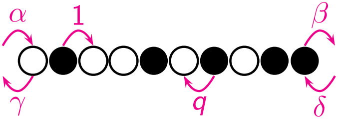
(C. and Williams 06)

Markov chain on Permutation tableaux

(C. and Williams 07)

Ongoing work

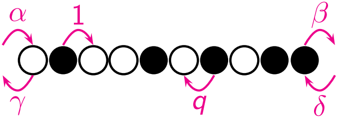
- General PASEP with γ and δ



New tableaux. Enumeration problems? Crossings? Combinatorics of Askey-Wilson polynomials? Grassmanians? (C. and Williams 09)

Ongoing work

- General PASEP with γ and δ

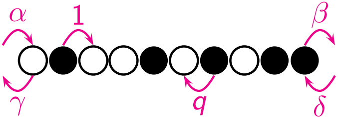


  New tableaux.

- Total positivity for cominuscule Grassmannians (Lam and Williams 08). Nice enumeration problems for Type B permutation tableaux. (C., Kim, Williams 09)

Ongoing work

- General PASEP with γ and δ

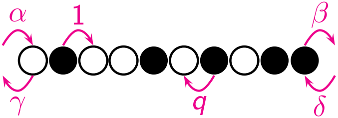


New tableaux.

- Nice enumeration problems for Type B permutation tableaux. (C., Kim, Williams 09)
- Total positivity for affine Grassmannians (Lam and Postnikov 09). Combinatorial setting : balanced graphs. Tableaux?

Ongoing work

- General PASEP with γ and δ



  New tableaux.

- Nice enumeration problems for Type B permutation tableaux. (C., Kim, Williams 09)
- Combinatorial setting : balanced graphs. Tableaux?
- PASEP with several types of particles and Koornwinder polynomials? (Haiman 07)

Thank you for your attention

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