Pancake sorting

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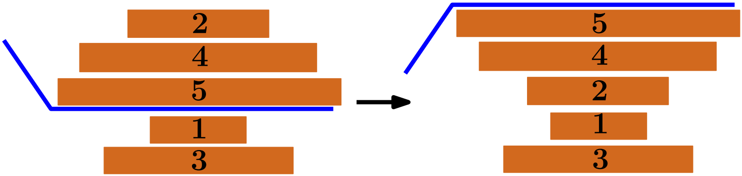
Definitions

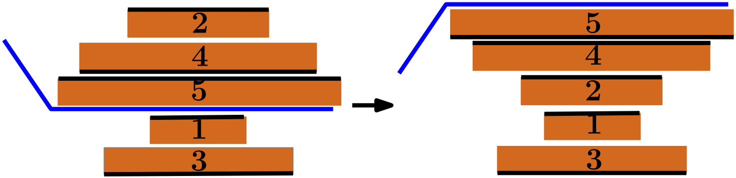
- Given: stack C of pancakes of different sizes and a spatula
- Aim: sort C by as few flips as possible . . . $f(C)$ steps
- $f(n)$ . . . minimum number of flips to sort any stack of size n



- Observation: $2n$ flips suffice

Burnt Version

- Each pancake burnt on one side
- Aim: stack $I_n :=$ sorted stack with burnt sides down
- $g(B)$, $g(n)$
- $-B$ ... stack B with all orientations changed
- $-I_n$ ... sorted stack with burnt sides up



- Observation: $3n$ flips suffice

Definitions

- Adjacency
  - Pancakes i and $i+1$ placed next to each other
  - Orientations of burnt pancakes must be the same as in I_n or in flipped I_n
  - Simplification: 1 and n can form an adjacency
- Block
  - maximal sequence or adjacent pancakes
  - can be contracted to a single burnt pancake
- Anti-adjacency (burnt version only) . . . adjacency in $-B$
- Clan (burnt version only) . . . block in $-B$
- $\mathbb{U}_n^m$ ($\mathbb{B}_n^m$) . . . set of stacks of n (burnt) pancakes with optimal sorting sequence of length m

Small stacks

n	$f(n)$		$g(n)$		$g(-I_n)$	
2	1	Garey et al. 1977	4	Cohen, Blum 1995	4	Cohen, Blum 1995
3	3	Garey et al. 1977	6	Cohen, Blum 1995	6	Cohen, Blum 1995
4	4	Garey et al. 1977	8	Cohen, Blum 1995	8	Cohen, Blum 1995
5	5	Garey et al. 1977	10	Cohen, Blum 1995	10	Cohen, Blum 1995
6	7	Garey et al. 1977	12	Cohen, Blum 1995	12	Cohen, Blum 1995
7	8	Garey et al. 1977	14	Cohen, Blum 1995	14	Cohen, Blum 1995
8	9	Robbins 1979	15	Cohen, Blum 1995	15	Cohen, Blum 1995
9	10	Robbins 1979	17	Cohen, Blum 1995	17	Cohen, Blum 1995
10	11	Cohen, Blum 1995	18	Cohen, Blum 1995	18	Cohen, Blum 1995
11	13	Cohen, Blum 1995	19	Korf 2008	19	Cohen, Blum 1995
12	14	Heydari, Sudborough 1997	21	Korf 2008	21	Cohen, Blum 1995
13	15	Heydari, Sudborough 1997	**22**		22	Cohen, Blum 1995
14	16	Kounoike et al. 2005	**23**		23	Cohen, Blum 1995
15	17	Kounoike et al. 2005	**25**		24	Cohen, Blum 1995
16	18	Asai et al. 2006	**26**		26	Cohen, Blum 1995
17	19	Asai et al. 2006	**28**		28	Cohen, Blum 1995
18	**20**				29	Cohen, Blum 1995
19	**22**				**30**	
20					**32**	
					$\left\lfloor \frac{3n+3}{2} \right\rfloor$	for $n \equiv 3 \pmod 4$

Computing values

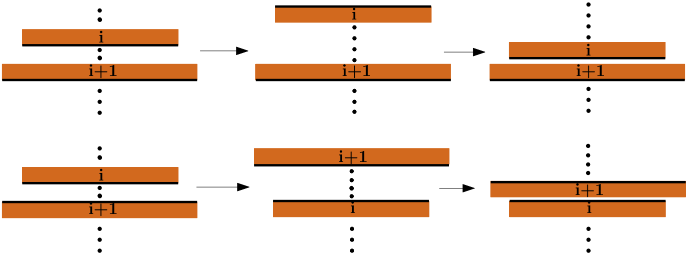
- exhaustive search - BFS
  - memory consuming
  - 2008 Korf - up to $f(15)$ and $g(12)$ - 457 GB of space
  - disk instead of RAM
- evaluate only hardest stacks:
  $\mathbb{U}_{18}^{21}$ determined from $\mathbb{U}_{17}^{m}$ with $m \geq 19 \ldots \mathbb{U}_{10}^{m}$, $m \geq 5$
- value of $f(19)$ computed on a grid of computers
  - Metacentrum `meta.cesnet.cz`
  - would take 5 years on a modern dual-core processor

Computing values - burnt case

Lemma (Cohen, Blum 1995)

In any stack B of burnt pancakes

- *Either a pair of pancakes can be joined in 2 flips*
- *or* $B = -I_n$



- $\mathbb{B}_{17}^{29}$ determined from $\{-I_{17}\} \cup \mathbb{B}_{16}^{m}$, where $m \geq 27 \ldots$

Known bounds

W.Gates, C.H.Papadimitriou 1979	$f(n)$	$g(n)$
Lower	$(17/16)n$	$(3/2)n-1$
Upper	$(5/3)(n+1)$	$2n+3$

Current	$f(n)$		$g(n)$		$g(-I_n)$	
Lower	$15\lfloor\frac{n}{14}\rfloor$	[HS]	$\lfloor\frac{3n+3}{2}\rfloor$		$\lfloor\frac{3n+3}{2}\rfloor$	
Upper	$\frac{18}{11}n+O(1)$	[CFMMSSV]	$2n-2$	[CB]	$\frac{3}{2}n+O(1)$	[HS]

CFMMSSV B.Chitturi, W.Fahle, Z.Meng, L.Morales, C.O.Shields, I.H.Sudborough, W.Voit, 2008

CB D.S.Cohen, M.Blum 1995

HS M.H.Heydari, I.H.Sudborough 1997

Lower bound on $g(B)$

$$v(B) := \Delta a(B) - \frac{1}{3}\Delta b(B) + \frac{1}{3}\Delta o(B) + \Delta I(B) + \frac{1}{3}\Delta II(B),$$

$a(B) :=$ number of adjacencies in B

$b(B) :=$ number of deep blocks (i. e. not on top)

$I(B) :=$ 1 if the largest pancake is placed correctly, 0 otherwise

$II(B) :=$ 1 if the two largest pancakes are placed correctly, 0 otherwise

$o(B) :=$ 1 if the smallest pancake is in a block or on top with burnt side up, 0 otherwise

$$x^{-}(B) := x(-B)$$

$$\Delta x(B) := x(B) - x^{-}(B)$$

Lemma

A single flip changes v by at most $4/3$.
Therefore B needs at least $\left\lceil \frac{3}{4}(v(I_n) - v(B)) \right\rceil$ flips to sort.

The algorithms of Gates and Papadimitriou

- Experimental results:
  - unburnt: $\approx 1.24n$
  - burnt: $\approx 1.5n$
- Explanation: From a random burnt stack, a join can be created in 1.5 flips on average
- But the resulting stack is not random

New algorithms

Theorem

There are algorithms that sort

- *a random stack of n unburnt pancakes in $17n/12 + O(1)$ flips on average*
- *a random stack of n burnt pancakes in $7n/4 + O(1)$ flips on average*

Conjecture

The average number of flips of the optimal algorithm for sorting burnt pancakes is

$$n + \Theta\left(\frac{n}{\log n}\right).$$