# Arc Consistency and Friends (joint work with Berit Grussien, Manuel Bodirsky, and Victor Dalmau)

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# Constraint Satisfaction

- The *constraint satisfaction problem* (CSP) is a general search problem
- In an instance of the CSP:
  - We are given: set of variables $A$, set $B$ called *domain*, constraints
  - Want to decide: does there exist assignment $A \to B$ satisfying all constraints?
- Many combinatorial problems can be naturally cast as cases of the CSP:
  - boolean satisfiability ($A =$ variables, $B = \{0, 1\}$)
  - graph coloring ($A =$ vertices, $B =$ colors)
  - scheduling ($A =$ events, $B =$ time slots)
- Complexity: NP-complete in general (in usual formulations)
- We will cast the CSP as the problem of deciding if there is a *homomorphism* between two given *relational structures*

# Structures and Homomorphisms

- A *signature* is a set of relation symbols $\sigma = \{R, \dots, \}$; each symbol $R$ has an associated *arity* $k_R$
- Example: to talk about graphs, signature $\sigma = \{E\}$ with $k_E = 2$
- A *structure* $\mathbf{B}$ over $\sigma$ consists of:
  - A *universe* $B$, a set (finite here)
  - A relation $R^{\mathbf{B}} \subseteq B^{k_R}$ for each $R \in \sigma$
- Given two structures $\mathbf{A}, \mathbf{B}$ over the same signature $\sigma$, a homomorphism is a mapping $h : A \to B$ such that

  $$\text{for each tuple } (a_1, \dots, a_k) \in R^{\mathbf{A}}, \\ \text{it holds that } (h(a_1), \dots, h(a_k)) \in R^{\mathbf{B}}$$

  (over all symbols $R \in \sigma$)
- Warning: I will mix terminology a bit (e.g. refer to elements of $A$ as *variables*)

# A Framework of Problems

- Official definition: CSP is to decide, given two structures $\mathbf{A}, \mathbf{B}$, if there exists hom. $\mathbf{A} \to \mathbf{B}$
- A heavily studied restriction of the CSP:

$$\mathrm{CSP}(\mathbf{B}) = \{\mathbf{A} \mid \text{exists hom. } \mathbf{A} \to \mathbf{B}\}$$

- Now, for each structure $\mathbf{B}$ we have a different computational problem $\mathrm{CSP}(\mathbf{B})$

# A Framework of Problems

- This restriction of CSP allows one to capture & place into a unified framework many different cases of CSP that have been studied:
  - 2-SAT
  - Horn-SAT
  - 3-Colorability
  - $H$-Colorability
  - Solving systems of equations
- Research problem: classify the complexity of all problems CSP($\mathbf{B}$)
- First results by Schaefer '78 (two-elt. structures), Hell & Nesetril '90 (undirected graphs)
- Another type of problem: given an algorithm (or algorithmic technique), describe the structures

$$\{\mathbf{B} \mid \text{CSP}(\mathbf{B}) \text{ solvable by algorithm } \}$$

# Arc Consistency

- Efficient (polytime) algorithm for detecting unsat. instances
- Idea: make local inferences using one variable at a time
- Generalizes *unit propagation* for CNF-SAT
- Basic reasoning technique in constraint solving, used in practice
- Theme of this talk:
  show how (in the CSP($\mathbf{B}$) world) arc consistency and extensions thereof can be studied and understood using algebraic tools

# Arc Consistency: the algorithm

- Given an instance $(\mathbf{A}, \mathbf{B})$ of the hom. problem...
- To each variable $a \in A$, associate a set $S_a$
- We maintain invariant: for any hom. $h : \mathbf{A} \to \mathbf{B}$ and any $a \in A$, it must hold that $h(a) \in S_a$
- Initialize $S_a = B$ for each $a \in A$
- Iterate until convergence:
  for each tuple $(a_1, \ldots, a_k) \in R^{\mathbf{A}}$:
  let $T = R^{\mathbf{B}} \cap (S_{a_1} \times \ldots \times S_{a_k})$ then, for each $i$, set $S_{a_i} = \pi_i(T)$
- If for some $a \in A$ it holds that $S_a = \emptyset$, output "unsatisfiable"
  Else output "?"

# A sound but incomplete procedure

- AC is "sound but incomplete"
  - If AC outputs "unsatisfiable" $\Rightarrow$ instance really unsat.
  - If AC outputs "?" $\Rightarrow$ instance may be sat. or unsat.
- Incompleteness follows from P not equal NP (as AC runs in P)
- A quick unconditional proof:
  - Let $\mathbf{B}$ be $K_3$
  - We have $E^{\mathbf{B}} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$
  - The projection of $E^{\mathbf{B}}$ onto either coordinate always gives $\{1,2,3\}$, so no inference will take place: algorithm terminates with $S_a = \{1,2,3\}$ for all $a \in A$
  - Let $\mathbf{A}$ be any graph with no hom. to $\mathbf{B}$, e.g. $\mathbf{A} = K_4$
  - On $(\mathbf{A},\mathbf{B})$, AC algorithm returns "?" on an unsat. instance

# The power structure

- We can give an algebraic condition that holds when AC algorithm returns "?"
- This will allow us to study AC from a number of angles using algebraic tools
- We can define, for each structure $\mathbf{B}$, a *power structure* $\mathcal{P}(\mathbf{B})$ such that

$$\text{AC returns "?" on } (\mathbf{A}, \mathbf{B}) \Leftrightarrow \text{there is a hom. } \mathbf{A} \to \mathcal{P}(\mathbf{B})$$

- Definition of $\mathcal{P}(\mathbf{B})$:
  - Universe is $\mathcal{P}(B) \setminus \{\emptyset\}$
  - For each $R$, we have $R^{\mathcal{P}(\mathbf{B})} = \{(\pi_1(S), \ldots, \pi_k(S)) \mid S \subseteq R^{\mathbf{B}}, S \neq \emptyset\}$
- Key point: from $\mathbf{B}$ there is a derived structure $\mathcal{P}(\mathbf{B})$ that characterizes AC in the above way

# A characterization of when AC works

- Although AC is not complete in general, there are some structures $\mathbf{B}$ where AC solves ("is complete for") CSP($\mathbf{B}$): for all structures $\mathbf{A}$

  there is a hom. $\mathbf{A} \to \mathcal{P}(\mathbf{B}) \Rightarrow$ there is a hom. $\mathbf{A} \to \mathbf{B}$
- Theorem (Feder & Vardi '93, Dalmau & Pearson '99): AC solves CSP($\mathbf{B}$) if and only if there is a hom. from $\mathcal{P}(\mathbf{B}) \to \mathbf{B}$
- Proof ($\Rightarrow$):
  - Suppose AC solves CSP($\mathbf{B}$).
  - There is a hom. $\mathcal{P}(\mathbf{B}) \to \mathcal{P}(\mathbf{B})$, hence there is a hom. $\mathcal{P}(\mathbf{B}) \to \mathbf{B}$
- Proof ($\Leftarrow$):
  - Suppose there is a hom. $\mathcal{P}(\mathbf{B}) \to \mathbf{B}$
  - Let $\mathbf{A}$ be such that there is a hom. $\mathbf{A} \to \mathcal{P}(\mathbf{B})$
  - Compose the two homs. to get a hom. $\mathbf{A} \to \mathbf{B}$

# What can we do with this characterization?

- Theorem (Feder & Vardi '93, Dalmau & Pearson '99): AC solves CSP($\mathbf{B}$) if and only if there is a hom. from $\mathcal{P}(\mathbf{B}) \to \mathbf{B}$
- What can we do with this characterization (and others like it)?
    1. Characterization implies that given a structure $\mathbf{B}$, it is *decidable* if AC solves CSP($\mathbf{B}$): simply check for a hom. from $\mathcal{P}(\mathbf{B}) \to \mathbf{B}$
       $\Rightarrow$ helps us understand which structures are solvable by AC
    2. Also helps us understand the class of structures $\{ \mathbf{B} \mid \text{CSP}(\mathbf{B}) \text{ solvable by AC} \}$ in aggregate: can prove
        - $\mathbf{B}$ solvable by AC, $\mathbf{B}'$ is an expansion of $\mathbf{B}$ by pp-def. relations
          $\Rightarrow$ $\mathbf{B}'$ solvable by AC
        - $\mathbf{B}$ solvable by AC, $\mathbf{B}'$ is hom. equivalent to $\mathbf{B}$
          $\Rightarrow$ $\mathbf{B}'$ solvable by AC

# Look-Ahead Arc Consistency (LAAC)

- LAAC algorithm:
  - Arbitrarily pick a variable $a$
  - Try to find a value $b$ such that after $a$ set to $b$, AC returns "?"
  - If no such value found, terminate and output "?"
  - If such a value found, fix $a$ to $b$, eliminate $a$, and repeat
- Algorithm either outputs "?" or a satisfying assignment
  Note: "one-sided error", but on the other side
- Only conceptual primitive used other than picking/setting variables is AC!
- Theorem (Chen & Dalmau '04): LAAC solves CSP($\mathbf{B}$) if and only if there is a hom. $h : \mathcal{P}(\mathbf{B}) \times \mathbf{B} \rightarrow \mathbf{B}$ such that $h(\{b\}, c) = b$ for all $b, c \in B$
- Notes: can decide if LAAC solves CSP($\mathbf{B}$);
  LAAC solves 2-SAT

# Peek Arc Consistency (PAC)

- Idea: take a peek at each variable
- PAC algorithm:
  - For each variable-value pair $(a, b)$, set $a$ to $b$ and check to see what AC returns
  - If for some variable $a$ it holds that all values $b$ result in AC returning "unsat", return "unsat"
  - Else return "?"
- Algorithm returns "unsat" or "?" as AC does
- Theorem (Bodirsky and Chen): Let $I(\mathcal{P}(\mathbf{B})^n)$ denote the induced substructure of $\mathcal{P}(\mathbf{B})^n$ containing all elements of $\mathcal{P}(\mathbf{B})^n$ with at least one coordinate a singleton.
  PAC solves CSP($\mathbf{B}$) $\Leftrightarrow \forall n \geq 1$, there's a hom. $I(\mathcal{P}(\mathbf{B})^n) \rightarrow \mathbf{B}$
- Note: induced substructure of a structure defined similarly to induced subgraph of a graph

# Peek Arc Consistency (PAC)

- Theorem:
  PAC solves CSP($\mathbf{B}$) $\Leftrightarrow \forall n \geq 1$, there's a hom. $I(\mathcal{P}(\mathbf{B})^n) \rightarrow \mathbf{B}$
- Proof ($\Leftarrow$):
  - Easier direction: have algebraic char., want to show alg. works
  - Suppose PAC doesn't return unsat. on $(\mathbf{A}, \mathbf{B})$
  - Then for each $a \in A$, have a hom. $\mathbf{A} \rightarrow \mathcal{P}(\mathbf{B})$ sending $a$ to singleton
  - Combining these homs., get hom. $\mathbf{A} \rightarrow \mathcal{P}(\mathbf{B})^{|A|}$ setting each $a \in A$ to a tuple having some singleton
  - This is a hom. $\mathbf{A} \rightarrow I(\mathcal{P}(\mathbf{B})^{|A|})$
  - Compose it with the hom. in theorem statement to get hom. $\mathbf{A} \rightarrow \mathbf{B}$

# Peek Arc Consistency (PAC)

- Theorem:
  PAC solves CSP($\mathbf{B}$) $\Leftrightarrow \forall n \geq 1$, there's a hom. $I(\mathcal{P}(\mathbf{B})^n) \to \mathbf{B}$
- Proof ($\Rightarrow$):
  - Suppose PAC solves CSP($\mathbf{B}$)
  - Want to show that there is a hom. $I(\mathcal{P}(\mathbf{B})^n) \to \mathbf{B}$
  - Consider input $\mathbf{A} = I(\mathcal{P}(\mathbf{B})^n)$; we show that PAC returns "?" on this
  - Let $a = (S_1, \ldots, S_n)$ be any variable from $A$
  - Know that by def., some $S_i$ is a singleton $\{b\}$
  - Consider the mapping $\pi_i$ that projects onto the $i$th coordinate
  - This is a hom. from $\mathbf{A}$ to $\mathcal{P}(\mathbf{B})$ sending $a$ to a singleton $\{b\}$
  - Hence, PAC returns "?" on $(\mathbf{A}, \mathbf{B})$
  - By assumption that PAC solves CSP($\mathbf{B}$), get hom. $\mathbf{A} \to \mathbf{B}$

# Singleton Arc Consistency (SAC)

- Idea: like PAC, but perform propagation
- SAC algorithm:
  - Loop until no more changes:
    - For each variable-value pair $(a, b)$, set $a$ to $b$ and run AC; if AC returns "unsat" then remove all instances of $(a, b)$ from instance
  - If for some variable $a$ there are no values $b$, return "unsat"; else return "?"
- Algorithm returns "unsat" or "?" as AC and PAC do
- Theorem: SAC solves CSP($\mathbf{B}$) $\Leftrightarrow \forall n \geq 1$, there's a hom. from ... $\rightarrow \mathbf{B}$
- Tractability results (in progress): SAC solves structures preserved by
  - any majority operation
  - any 2-semilattice operation

# A hierarchy

- Theorem: AC $\subsetneq$ LAAC $\subseteq$ PAC $\subsetneq$ SAC
- Here, an algorithm denotes the set of structures **B** that it solves
- Note: proof of inclusion LAAC $\subseteq$ PAC is algebraic, uses algebraic characterizations of algorithms
  - I know of no proof based just on the algorithm descriptions

# Perspective

- Algebraic tools have been used to understand the family of problems CSP(**B**)
- Work on understanding when problems fall in/out of complexity classes:
  - Sufficient condition for NP-hardness - Bulatov et al.
  - Sufficient conditions for hardness for other comp. classes - Larose & Tesson
  - Classification results for large families of structures - Bulatov
- Work on understanding when certain algorithmic techniques work:
  - Maintaining a succinct representation / "few subalgebras" - Dalmau, Berman et al.
  - Bounded width - Larose & Zadori, Atserias et al., Bulatov, Kiss & Valeriote, Barto & Kozik, ...
- In contrast, here, we use algebraic tools to understand *particular, concrete* algorithms

# Open questions

- Bounded width algorithms $\approx$ generalizations of arc consistency where inference performed on $k$ variables at a time
- Open question 1: are there analogs of the structure $\mathcal{P}(\mathbf{B})$ that characterize higher forms of consistency?
- Class of structures $\mathbf{B}$ for which bounded width solves CSP($\mathbf{B}$) has been classified (Barto & Kozik)
- Open question 2: can singleton arc consistency solve all such problems?
- Thanks!