Problems and Results in Asymptotic Combinatorics

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Thank-you to Coauthors

Part I, Macmahon statistic: Svante Janson & Doron Zeilberger

Part II, Durfee polynomials: Sylvie Corteel & Carla Savage

Part III, Prescribed parts and multiplicities: Herb Wilf

Smoothness

A sequence a_n is *unimodal* provided for some K

$$a_1 < a_2 < \cdots < a_K \geq a_{K+1} \geq \cdots$$

log concave:

$$(a_n)^2 \geq a_{n-1}a_{n+1}$$

Proposition: If $a_n > 0$ then log concavity implies unimodality.

A related Matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ & & \vdots & & \end{pmatrix}$$

a_n is *totally positive* if all sub-matrices have nonnegative determinant

Theorem: a_n is t.p. $\iff \sum_n a_n x^n$ has all roots in $(-\infty, 0]$

Gaussian Polynomials

$$\binom{a+b}{a}_q = \prod_{j=1}^{a} \frac{1-q^{b+j}}{1-q^j}$$

Combinatorics:

$$[q^n]\binom{a+b}{a}_q$$

is the number of partitions of n with all parts $\leq a$, and no more than b parts.

Polynomial ? Positive coefficients ?

$$\binom{N}{a}_q = q^a\binom{N-1}{a}_q + \binom{N-1}{a-1}_q$$

Answers

Totally Positive: obviously not, roots on unit circle

Log concave: obviously not, since ($a, b \geq 2$)

$$\binom{a+b}{a}_q = 1 + q + 2q^2 + \cdots$$

Unimodal: yes, Sylvester (1878); and O'Hara (1990)

But Wait

Consider $c_j = [q^j]\binom{2n}{n}_q$

$j \in \{m-1, m, m+1\}$ $\qquad m = n^2/2 - 1$

n	$(c_m)^2 - c_{m+1} \times c_{m-1}$
2	-1
4	-7
6	-165
8	-1529
10	44160
12	7715737
14	905559058
16	101507214165
18	11955335854893
20	1501943866215277

Central Limit Theorem

$$\binom{a+b}{a}_q = c_0 + c_1 q + c_2 q^2 + \cdots$$

The numbers c_j, normalized, determine a mean μ and a variance σ^2

$$\sup_x \left| \binom{a+b}{a}^{-1} \sum_{j \le \mu + x\sigma} c_j \;-\; \frac{1}{2\pi} \int_{-\infty}^{x} e^{-t^2/2} dt \right| \;\rightarrow\; 0$$

as $a, b \rightarrow \infty$.

Central versus Local

"If one can prove a central limit theorem for a sequence $a_n(k)$ of numbers arising in enumeration, then one has a qualitative feel for their behavior. A local limit theorem is better because it provides asymptotic information about $a_n(k)$ $\ldots$,"

Bender, 1973

Usual way to pass from Central to Local: unimodality (misses center); or log-concavity

A Local Limit Theorem

Theorem.

$$[q^n]\binom{a_1+\cdots+a_K}{a_1,\ldots,a_K}_q \;=\; \frac{1}{\sigma\sqrt{2\pi}}\binom{a_1+\cdots+a_K}{a_1,\ldots,a_K}$$

$$\times\;\left(e^{-x^2/2}+O(\frac{1}{m})\right)$$

where

$$n=\mu+x\sigma$$

$$m=\min(a_1,\ldots,a_K)$$

$$\max\{a_1,\ldots,a_K\}=O(e^{(c-\delta)Km}).$$

Durfee Square

$$24 = 10 + 6 + 3 + 3 + 1 + 1$$

D	D	D	x	x	x	x	x	x	x
D	D	D	x	x	x				
D	D	D							
x	x	x							
x									
x									

Ferrer’s diagram, with Durfee square D’ed

GF via Durfee

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2 \cdots (1-x^d)^2}$$

$$p(n) = \#\ \text{partitions of the integer } n$$

Durfee Polynomials

$$\sum_{d=0}^{\infty} \frac{y^d x^{d^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^d)^2}$$

$$= \sum_{n=0}^{\infty} x^n \left[\sum_{d=0}^{\lfloor n^{1/2} \rfloor} p(n,d) y^d \right]$$

$p(n,d) = \#$ partitions of the integer n with Durfee square size d

Recursion

$$p(n,d) = 2p(n-d,d) + p(n-2d+1,d-1) - p(n-2d,d)$$

Can be obtained from the GF, or combinatorially

Table

$p(n, d)$

$n \backslash d$	1	2	3	Total
1	1			1
2	2			2
3	3			3
4	4	1		5
5	5	2		7
6	6	5		11
7	7	8		15
8	8	14		22
9	9	20	1	30
10	10	30	2	42

Asymptotics

$$p(n,d) = \sum_{n_1+n_2=(n-d^2)} P(n_1,d)P(n_2,d)$$

$$P(n,k) = \#\ \text{partitions of } n \text{ with } \le k \text{ parts}$$

In a series of papers from the early 1950's, George Szekeres has obtained an asymptotic series for $P(n,k)$.

Theorem

Uniformly for $\epsilon \leq x \leq 1 - \epsilon$

$$p(n, xn^{1/2}) = \frac{F(x)}{n^{5/4}} \exp\left\{n^{1/2}G(x) + O(n^{-1/2})\right\}$$

$$F(x) = \cdots$$

$$G(x) = \cdots$$

For Those Who Want to Know

$$F(x) = 2\pi^{1/2} f(u)^2 (2+u^2)^{5/4} \left(g(u) - ug'(u) - u^2 g''(u)\right)^{-1/2}$$

$$G(x) = 2g(u)(2+u^2)^{-1/2}$$

$$u = \sqrt{\frac{2x^2}{1-x^2}}$$

For Those Who REALLY Want to Know

$$f(u) = \frac{v}{2\pi u\sqrt{2}} \left(1 - e^{-v} - \frac{u^2 e^{-v}}{2}\right)^{-1/2}$$

$$g(u) = \frac{2v}{u} - u\log(1 - e^{-v})$$

$$u^2 = \frac{v^2}{\int_0^v \frac{t}{e^t - 1} dt}$$

Corollaries

1. The numbers $p(n,d)$, $0 \le d \le \lfloor n^{1/2} \rfloor$ are asymptotically normal as $n \to \infty$

$$\mu_n, \sigma_n^2 \sim c_1 n^{1/2}, c_2 n^{1/2}$$

2. For all n sufficiently large, and $\epsilon n^{1/2} \le d \le (1-\epsilon) n^{1/2}$

$$p(n,d)^2 \ge p(n,d+1)p(n,d-1)$$

3. For all n sufficiently large, the mean and the mode differ by less than 1

Getting to the Roots

The latter three findings would all be implied by

Conjecture: For all n, the Durfee polynomial $D_n(y) = \sum_d p(n,d)y^d$ has all its roots real and nonpositive.

Empirical Evidence

Theorem. For $n \leq 1000$ Durfee polynomial $D_n(y)$ has real roots only

Theorem: For $n \leq 5000$, the Durfee mean and mode differ by less than 1

The whole story:
erc, Corteel, & Savage
Durfee polynomials, *Electron. J. Combin.* **5** (1998), Research Paper 32

Questions: Asymptotic 3-positivity; $n = 1,000,000$

Restricted Parts

Let S be a set of positive integers, and $p_S(n)$ be the number of partitions of n all of whose parts lie in S.

Possible growth rates?

$$S = \{1, 2, 3, \ldots\} \qquad \log p_S(n) \sim C n^{1/2}, \quad C = \pi\sqrt{2/3}$$

$$S = \{1, 2, 4, 8, \ldots\} \qquad \log p_S(n) \sim C(\log n)^2, \quad C = (2\log 2)^{-1}$$

Credits: Hardy-Ramanujan & de Bruijn

A Theorem of Schur

An example of polynomial growth of $p_S(n)$
Assume $\gcd(a_1, \ldots, a_k) = 1$. Then,

$$S = \{a_1 < a_2 < \cdots < a_k\} \qquad \log p_S(n) \sim C \log n, \quad C = k - 1$$

More precisely,

$$p_S(n) \sim \frac{n^{k-1}}{(k-1)! a_1 a_2 \cdots a_k}$$

Remark: For any S (finite or infinite) $p_S(n)$ is positive for all sufficiently large n if and only if $gcd(S) = 1$

Multiplicites & Parts

Definition. Let S and M be two sets of positive integers, the allowable parts and their multiplicities. (Let $0 \in M$, too.)

$p(n; S, M)$ is the number of partitions $\lambda \vdash n$ into parts taken from the set S, and such that each part appearing in λ has multiplicity in M.

$$
\begin{aligned}
p(n; S, M) = \#\{ \quad & \text{pairs } (n_1, \ldots, n_k), (m_1, \ldots, m_k) : \\
& 1 \le n_1 < n_2 < \cdots < n_k \\
& n_i \in S, m_i \in M \\
& n = m_1 n_1 + \cdots m_k n_k \\
\}.
\end{aligned}
$$

Estimates of Growth

Let $M(x) = \#\{m \leq x : m \in M\}$

$$p(n;S,M) \leq \prod_{a_i \in S} M(n/a_i)$$

$$\exists r \leq n^2 s.t. p(r;S,M) \geq \frac{1}{n^2+1} \prod_{a_i \in S} M(n/a_i)$$

If $p(n;S,M)$ is monotone,

$$p(n;S,M) \geq \frac{1}{n+1} \prod_{a_i \in S} M(\sqrt{n}/a_i)$$

Slow, but not Polynomial

Theorem: For any infinite S and constant k

$$p_S(n) \neq O(n^k)$$

Theorem: For any $\omega(n) \to \infty$, there exists infinite S such that

$$p_S(n) = O(n^{\omega(n)})$$

Slow given M and S

Let $S = \{2^{2^j}\}_{j=0}^{\infty}$, and

$$M = \{0\} \cup \{2^{2^j}\}_{j=0}^{\infty}$$

Then

$$p(n; S, M) \leq (\log n)(\log\log n)^{\log\log n}$$

However, $p(n; S, M) = 0$ for many n

Challenge

Are there infinite sets S and M such that

$$p(n; S, M) > 0$$

for all sufficiently large n; yet,

$$p(n; S, M) = O(n^C)$$

Second Challenge

Sufficient conditions on infinite sets S and M to assure

$$p(n; S, M) > 0$$

for all sufficiently large n