Minimum Area Venn Diagrams

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Outline

Basic Definitions
Extending a Venn Diagram
Using Polyominoes

Minimum Area
Minimum Area Venn Diagram Definition
Drawing the PolyVenn

Expansion
Half Set Systems
Sufficient Conditions
PolyVenns for all $n$

Omiting the Empty Set

Summary
What We Know
What We Don’t Know
Basic Definitions

- A set of $n$ closed curves.
Basic Definitions

- A set of $n$ closed curves.
- Overlapping each other at single points.
Basic Definitions

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- Dividing the plane into $2^n$ regions.
Basic Definitions

- A set of $n$ closed curves.
- Overlapping each other at single points.
- Dividing the plane into $2^n$ regions.
- Each region is a unique subset of the interior of the curves.
Venn Diagram Examples

Venn diagrams as circles
Venn Diagram Examples

Venn diagrams as ellipses
Venn Diagram Examples

Venn diagrams as triangles

Thanks to Jeremy Carroll
Venn Diagram Examples

Venn diagrams with minimum vertices.
Venn Diagrams for any Number of Curves

We can take any Venn diagram and extend it to another one by finding a suitable curve.
Venn Diagrams for any Number of Curves

We can take any Venn diagram and extend it to another one by finding a suitable curve.

- It must pass through each existing region exactly once.
Venn Diagrams for any Number of Curves

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Polyominoes as Curves

We consider using polyominoes whose perimeters represent the curves.
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**Polyomino** A simply connected set of unit squares on the plane
Polyominoes as Curves

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**Domino** Two unit squares.
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**Polyomino**  A simply connected set of unit squares on the plane

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**Triomino**  Three unit squares.
Polyominoes as Curves

We consider using polyominoes whose perimeters represent the curves.

Polyomino  A simply connected set of unit squares on the plane

  Domino    Two unit squares.
  Triomino  Three unit squares.
  Tetromino Four unit squares.
Minimum Area

What would a Minimum Area Venn diagram using polyominoes look like?

- The perimeters of the polyominoes would intersect along horizontal or vertical units.
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Questions we answer:
- Can the polyomino Venn diagram (polyVenn) be confined to a rectangular grid?
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- Do polyVenns exist for all values of $n$?
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- Can the polyomino Venn diagram (polyVenn) be confined to a rectangular grid?
- If so, are there limits on the grid dimensions?
- Do polyVenns exist for all values of $n$?

YES.
PolyVenn Diagrams
## PolyVenn Diagrams

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### Explanation:
- **AB**: The first two letters indicate the sets involved.
- **BC**: The subsequent letters represent the Venn regions.
Two ways to discover the polyVenn

1. The heuristic method:
Two ways to discover the polyVenn

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   - Fill each of the $2^n$ subsets into the squares one by one.
Two ways to discover the polyVenn

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   - Fill each of the $2^n$ subsets into the squares one by one.
   - Watch for disconnected elements and *holes*.
Two ways to discover the polyVenn

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2. The algorithmic method:
   - Cover $2^{n-1}$ unit squares with a simply connected piece while overlapping each previous piece on $2^{n-2}$ squares.

Special thanks to Matt Klimesh who shared his algorithm and was the inspiration for this type of construction.
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- Blue squares represent units
- Pink squares represent the minimum area
- Yellow square represents the expansion
- Red square represents omitting the empty set
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Legend:
- Blue: Minimum Area
- Green: Expansion
- Yellow: Omitting the Empty Set
- Orange: Summary
Filling in the pieces
Filling in the pieces
Filling in the pieces
Filling in the pieces
Filling in the pieces
Filling in the pieces
The Half Set System

Definition

An $n$-HSS is a set $\{S_1, S_2, \ldots, S_n\}$ of subsets of $\{1, 2, \ldots, 2^n\}$ with the property that for any nonempty subset $A \subseteq \{1, 2, \ldots, n\}$

$$\left| \bigcap_{i \in A} S_i \right| = 2^{n-|A|}.$$
The Half Set System

Theorem

Let \( \{S_1, S_2, \ldots, S_n\} \) be a collection of subsets of \( \{1, 2, \ldots, 2^n\} \).

For all \( A \subseteq \{1, 2, \ldots, n\} \)

\[
\left| \bigcap_{i \in A} S_i \right| = 2^n - |A|
\]

if and only if for all subsets \( B \subseteq \{1, 2, \ldots n\} \), there is a unique element \( m \in \{1, 2, \ldots, 2^n\} \) such that

\[
\{m\} = \left( \bigcap_{i \in B} S_i \right) \cap \left( \bigcap_{i \notin B} \overline{S_i} \right).
\]
We can expand when...

Suppose we have a $2^r \times 2^c$ grid that holds an existing polyVenn. Then we can expand this polyVenn to create one on a $2^{r+r'} \times 2^{c+c'}$ grid if the following is true:

• There is a $(n' = r' + c')$ half set system for each of the $2^r \times 2^c$ original unit squares.

• The pieces of the half sets can be arranged so they are simply connected on the larger grid.
We can expand when...

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- There is a $(n' = r' + c')$ half set system for each of the $2^r \times 2^c$ original unit squares.
- The pieces of the half sets can be arranged so they are simply connected on the larger grid.
An example...

...of a disconnected piece that becomes connected in the larger piece.
Two By Two Expansion

Theorem

*If there is a polyVenn on a $2^r \times 2^c$ grid, then there is a polyVenn on a $2^{r+1} \times 2^{c+1}$ grid.*
Proof By Example

• Take an existing polyVenn.
Proof By Example

- Take an existing polyVenn.
- Double both its rows and columns.
Proof By Example

- We need two more pieces for the new polyVenn.
Proof By Example

- We need two more pieces for the new polyVenn.
Proof By Example

• We need two more pieces for the new polyVenn.
Proof By Example

- We need two more pieces for the new polyVenn.
Proof By Example

- We need two more pieces for the new polyVenn.
A Vertical Expansion Example
Two By Four Expansion

Theorem

*If there is a polyVenn on a $2^r \times 2^c$ grid, then there is a polyVenn on a $2^{r+1} \times 2^{c+2}$ grid.*
Two By Four Expansion Proof

- Suppose we want to expand a $2^r \times 2^c$ polyVenn.
Two By Four Expansion Proof

• Suppose we want to expand a $2^r \times 2^c$ polyVenn.
• And we had $r + c$ Half Set Systems for a $2 \times 4$ grid.
Two By Four Expansion Proof

- Suppose we want to expand a $2^r \times 2^c$ polyVenn.
- And we had $r + c$ Half Set Systems for a $2 \times 4$ grid.
  - We use variations on the same one.
Two By Four Expansion Proof

Expanding horizontally

[Diagram showing expansion process]
Two By Four Expansion Proof
Expanding horizontally
Two By Four Expansion Proof

Expanding horizontally
Two By Four Expansion Proof

Expanding horizontally

Diagram showing the expansion process step by step.
Two By Four Expansion Proof
Expanding horizontally
Two By Four Expansion Proof

Expanding horizontally
Two By Four Expansion Proof

Expanding horizontally
Two By Four Expansion Proof

Expanding vertically
Grids With $2^n - 1$ Unit Squares

- What about $h \times w$ grids where $2^n - 1 = hw$?
Grids With $2^n - 1$ Unit Squares

- What about $h \times w$ grids where $2^n - 1 = hw$?
- We know there is a polyVenn whenever $h = 2^{n/2} - 1$ and $w = 2^{n/2} + 1$. 
The $\left(2^r - 1\right) \times \left(2^r + 1\right)$ polyVenn
What We Know So Far

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**KEY:**
- **T** trivial
- **K** Klimesh discovery
- **B** Bultena discovery
- **X** not possible
- **U** unknown
The Limits We Know

Lemma

A single row polyVenn does not exist for any \( n > 2 \).
The Limits We Don’t Know

Conjecture

For each $r$, there is a value $c_r$ for which a polyVenn on a $2^r \times 2^{c_r}$ grid exists and for which no polyVenn exists for more than $2^{c_r}$ columns.
The Limits We Don’t Know

Conjecture

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Conjecture

The value of $c_1$ is 4. In particular, there is no polyVenn on a $2 \times 32$ grid.
Further Research

- Turn the LIMITS WE DON’T KNOW into the LIMITS WE KNOW.
- Convexity of Polyominoes
- Properties of the Perimeters
Acknowledgements

A special thanks to:

• Frank Ruskey
• Matthew Klimesh
• Stirling Chow
• Khalegh Ahmadi
Thank-you!