Colourings, Homomorphisms, and CSPs

CANADAM 2009

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Graph homomorphisms definition

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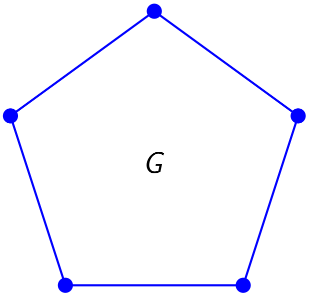
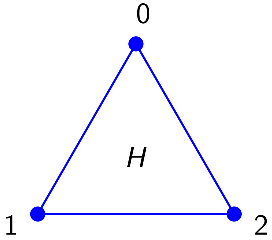
Adjacent vertices receive adjacent images.

We write $G \to H$ ($G \not\to H$) if there is a homomorphism (no homomorphism) of G to H.

An example





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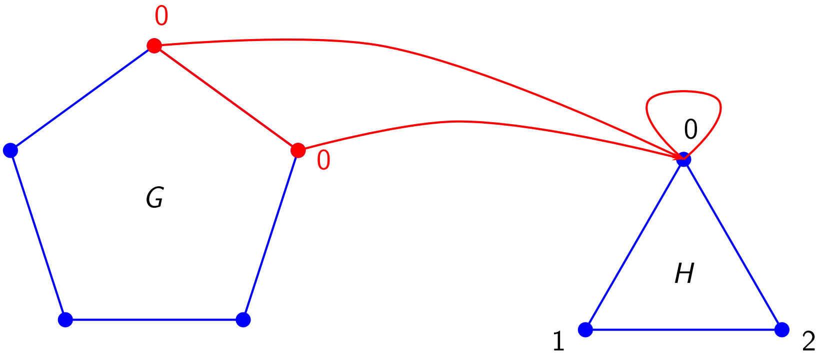
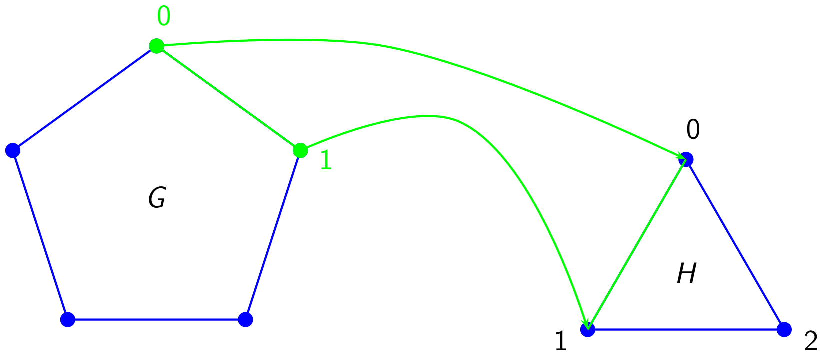


This assignment requires a loop on vertex 0 (in H)

An example



This assignment is allowed.

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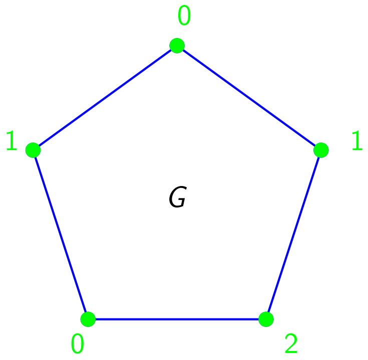
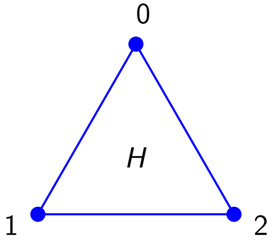
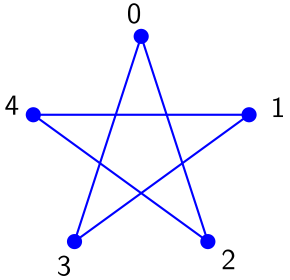




This labeling is a homomorphism $G \to H$.

A partitioning problem

G



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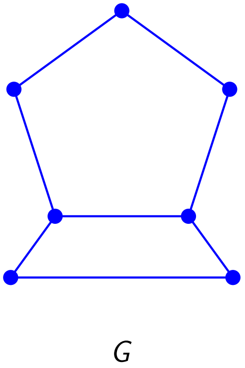

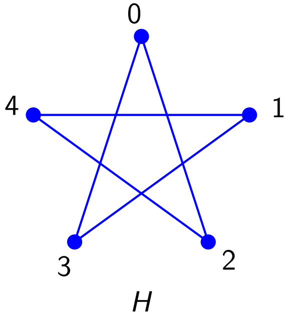

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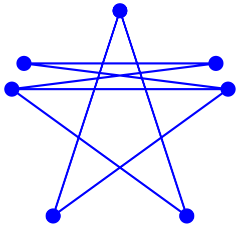

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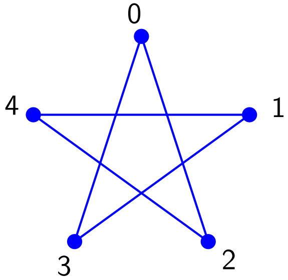

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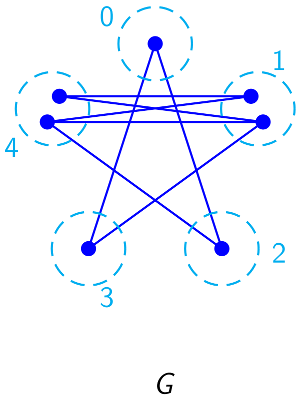

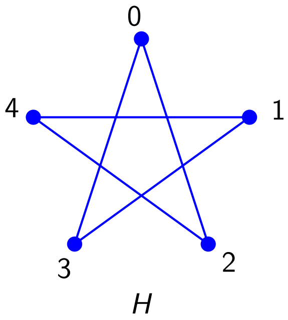

Homomorphisms Generalize Colourings

- $G \to K_n$ iff G is n-colourable.
- $G \to K_{p,q}$ iff G admits a (p, q)-circular colouring.
  $(uv \in E(K_{p,q})$ if $q \leq |u - v| \leq p - q)$

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- $G \to K(p, q)$ iff G admits a p/q-fractional colouring.
- Many colouring generalizations can be expressed in the language of homomorphisms with the appropriate *family of calibrating graphs*.

Labelling with a condition at distance 2

Definition (Griggs and Yeh)

Let G be a graph. A L(2,1) labelling of G is a function $\lambda : V(G) \to [n]$ such that

- $|\lambda(u) - \lambda(v)| \geq 2$ for $uv \in E$
- $|\lambda(u) - \lambda(v)| \geq 1$ for u and v at distance 2.

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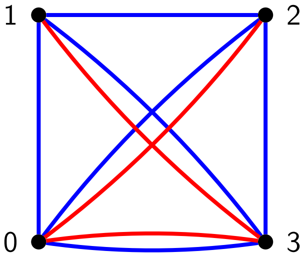

G is red

G^2 is blue

Hom preserves both relations

Did we gain anything?

- Edge-coloured homomorphisms too general (for me) to get much traction on $L(2,1)$-labellings.
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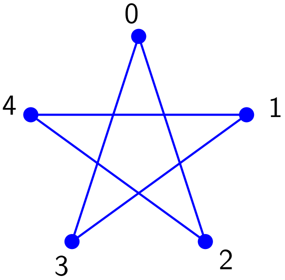
- Locally injective homomorphisms good tool for $L(2,1)$-labellings: Fiala, Kratochvíl, Pór
- Digraph versions in Nancy Clarke's talk.
- Ross Kang & Putra Manggala's talk deals with edge-colouring and a distance condition.

Assigning lists



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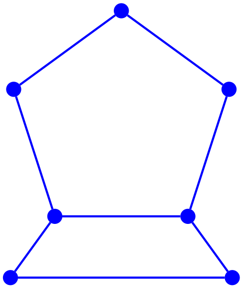

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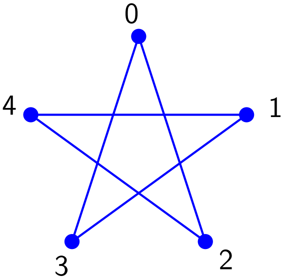

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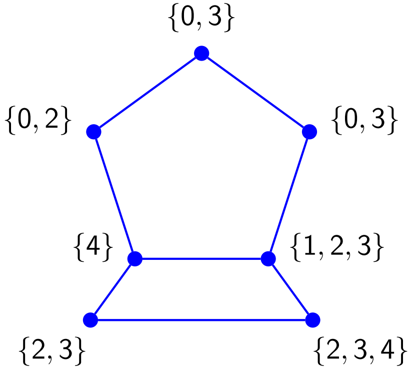

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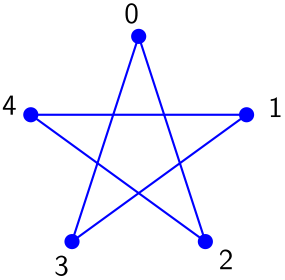

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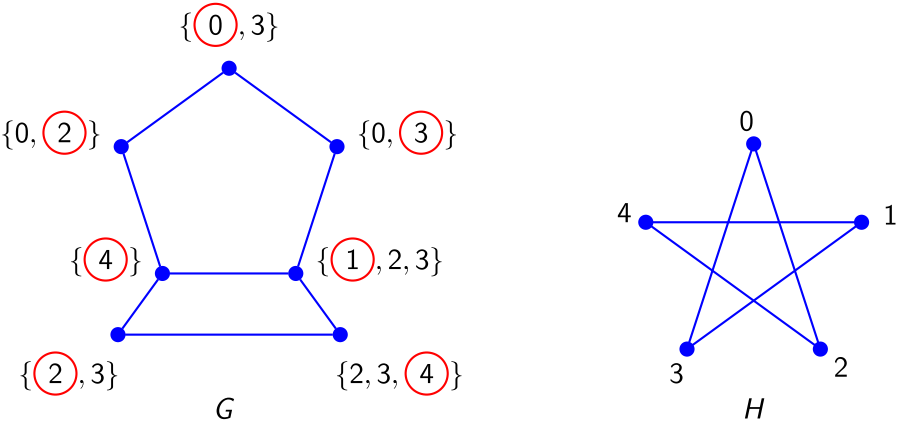

Definition of list homomorphism

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Let G and H be graphs. Suppose to each $v \in V(G)$ there is an assigned list $L(v) \subseteq V(H)$. A list homomorphism of G to H is a homomorphism $f : G \to H$ such that $f(v) \in L(v)$ for all v.

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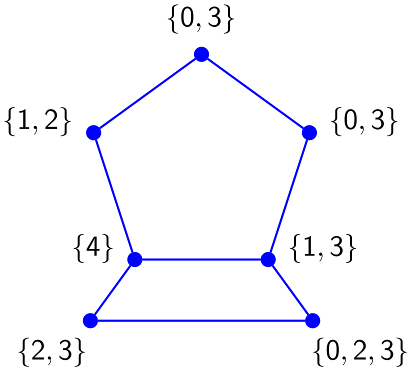
- $L(v) = V(H)$ for all v, is a standard homomorphism.
- Restrict images.
- Pre-colourings: lists have size one or $V(H)$.
- Choosability: lists have a fixed size.

Assigning lists



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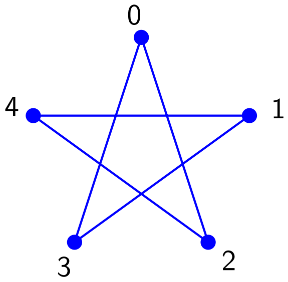

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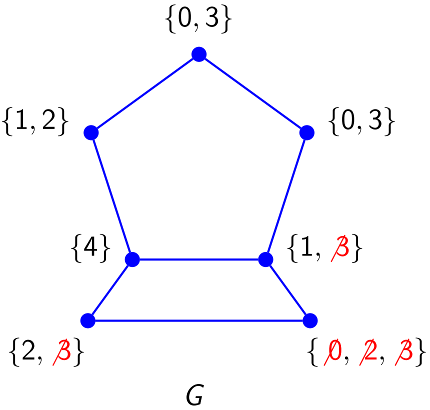

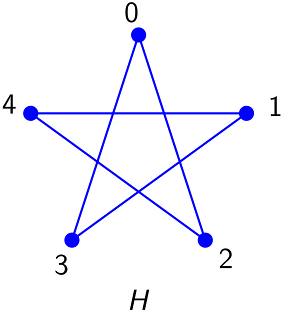

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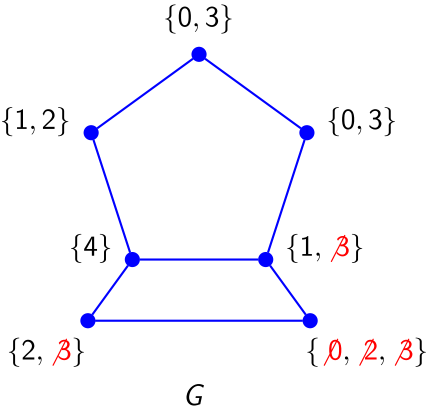

The edge-consistency check fails. $G \not\to H$.

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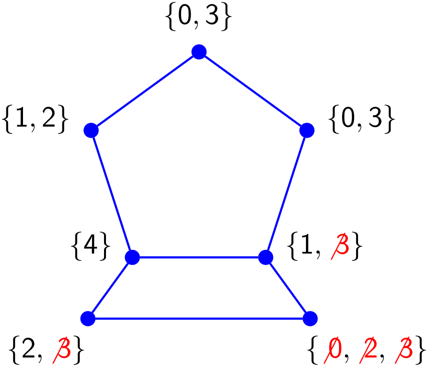

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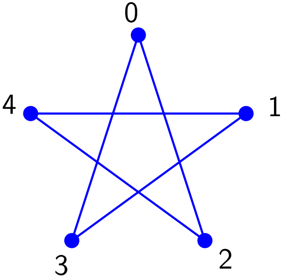

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Success of edge-consistency does not ensure $G \to H$.

Computational complexity of homomorphism problems

Definition

Let H be a fixed graph.

HOM-H
Instance: A graph G.
Question: Does G admit a homomorphism to H?

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Theorem (Hell and Nešetřil, 1990)

If H is bipartite or contains a loop, then HOM-H is polynomial time solvable; otherwise, HOM-H is NP-complete.

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Instance: A graph G and list $L(v)$.

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If H is a reflexive interval graph, then LIST HOM-H is polynomial time solvable; otherwise, LIST HOM-H is NP-complete.

Theorem (Feder, Hell, and Huang, 2003)

If H is a bi-arc graph, then LIST-HOM-H is polynomial time solvable; otherwise, LIST-HOM-H is NP-complete.

The X-property

Definition (Gutjahr, Welzl, Woeginger)

Let D be a digraph. An X-enumeration of the vertices is a total order with the property that $(a, b) \in E$ and $(c, d) \in E$ implies $(\min\{a, c\}, \min\{b, d\}) \in E$.

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Proposition

Suppose D is a digraph with an $\underline{X}$-enumeration of its vertices. Further suppose G is a digraph with lists L for which the consistency check succeeds. Then $G \to D$.

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Proof.

Take the minimum value in each list (after the consistency check). □

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- Near unanimity functions (NUF) generalize the concept of $\underline{\text{X}}$.
- Weak NUFs seem to be both the polynomial time *tool* and the dichotomy *classification*.
- Details to follow in Jacobus Swarts' talk.

Homomorphism Order

loop

$\vdots$

K_4

K_3

$\vdots$

K_2

K_1

Homomorphism Order

loop

$\vdots$

K_4

K_3

C_5

C_7

$\vdots$

K_2

K_1

Homomorphism Order

loop

⋮

K_4

K_3 Grötzsch $\cdots$

| ← A copy of the whole order

C_5

C_7

⋮

K_2

K_1

Homomorphism Order

loop

$\vdots$

$\mathbb{N} \hookrightarrow$

K_4

$\mathbb{Q} \hookrightarrow$ K_3 Grötzsch $\cdots$

$\longleftarrow$ A copy of the whole order

$\mathcal{P} \hookrightarrow$ C_5

C_7

$\vdots$

K_2

K_1

You can't paint yourself into a corner

Theorem (Albertson, 1998)

Suppose G is any planar graph and $W \subseteq V(G)$ such that the distance between any two vertices of W is at least 4. Any 5-colouring of the vertices of W can be extended to a 5-colouring of G.

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Theorem (Albertson, 1998)

Suppose $\chi(G) = r$ and $W \subseteq V(G)$ such that the distance between any two vertices of W is at least 4. Any $r+1$-colouring of the vertices of W can be extended to a $r+1$-colouring of G.

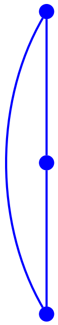
Extending circular colourings

Take the concept of extending a colouring by being allowed to move up in the homomorphism order.

Theorem (Albertson, West 2006)

Given k, d, k', d' with $k'/d' > k/d \geq 2$. Let $\ell = \lceil kk'/(2(k'd - kd')) \rceil$. If $\chi_c(G) \leq k/d$, and $P \subset V(G)$ is an independent set such that $d(P) \geq 2\ell$, then every precolouring of P from $\mathbb{Z}_{k'}$ extends to a (k', d')-colouring of G.

A uniquely 3-colourable graph



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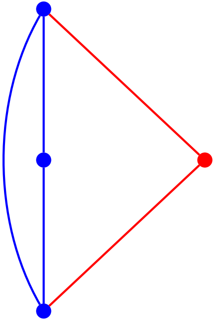

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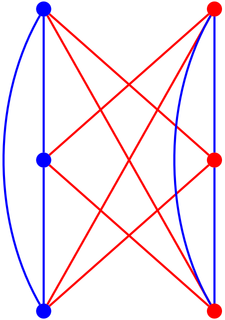

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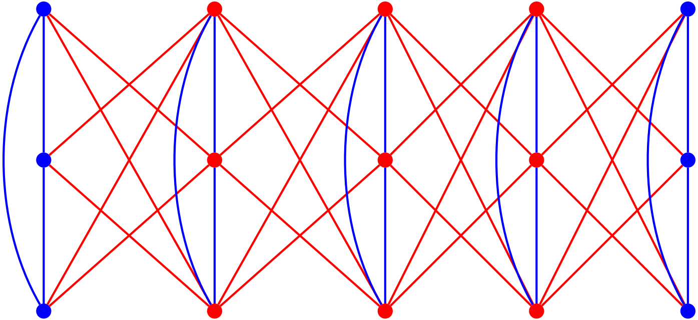

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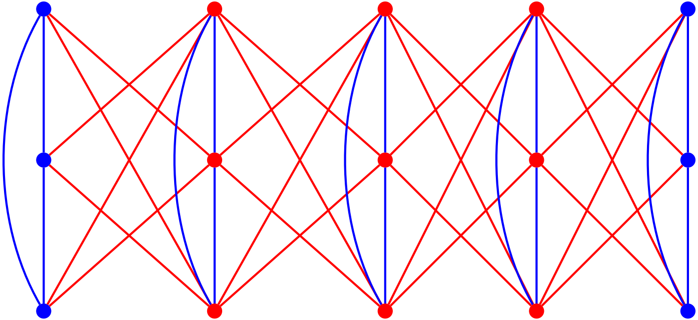

- The product $K_3 \bowtie P_5$.

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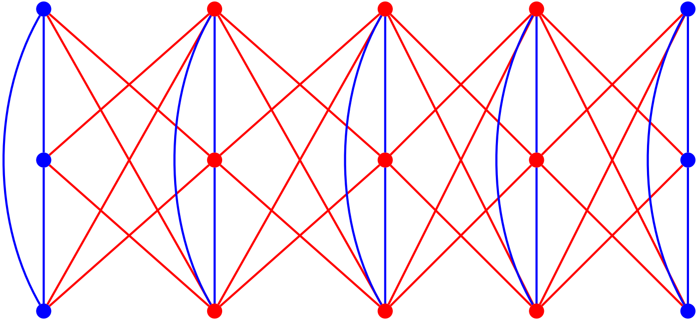

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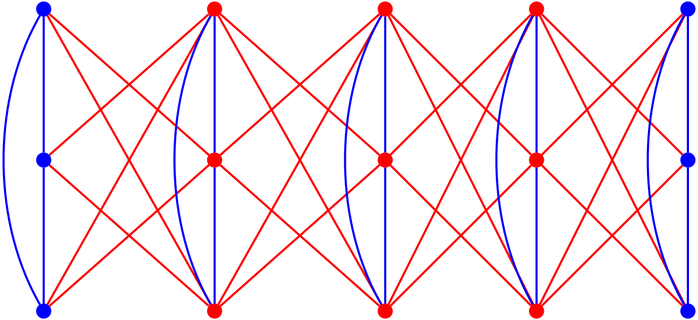

- The product $K_3 \bowtie P_5$.
- Ideas work for any core G replacing K_3.
- When we have an extra colour available, successive columns have nearly the same colouring. Ideas to follow in Ruth Haas' talk.

It comes down to attitude

We will modify γ so that it agrees with c on W. Suppose $u \in W$. If $\gamma(u) = c(u)$,

– Mike Albertson

It comes down to attitude

We will modify γ so that it agrees with c on W. Suppose $u \in W$. If $\gamma(u) = c(u)$, we love it.

– Mike Albertson