Leaf Powers: Recent Results and Open Problems

Andreas Brandstädt

(joint work with Christian Hundt, Van Bang Le, Federico Mancini, Dieter Rautenbach, R. Sritharan, and Peter Wagner)

Contents

Leaf Powers

[Nishimura, Ragde, Thilikos, On graph powers for leaf-labeled trees, *J. Algorithms* 2002]:

A finite undirected graph $G = (V,E)$ is a *k-leaf power* if there is a tree $T = (U, F)$ whose set of leaves is V such that for all $x,y \in V$

$$xy \in E \Leftrightarrow \mathrm{dist}_T(x,y) \leq k.$$

T is called a *k-leaf root of G*.

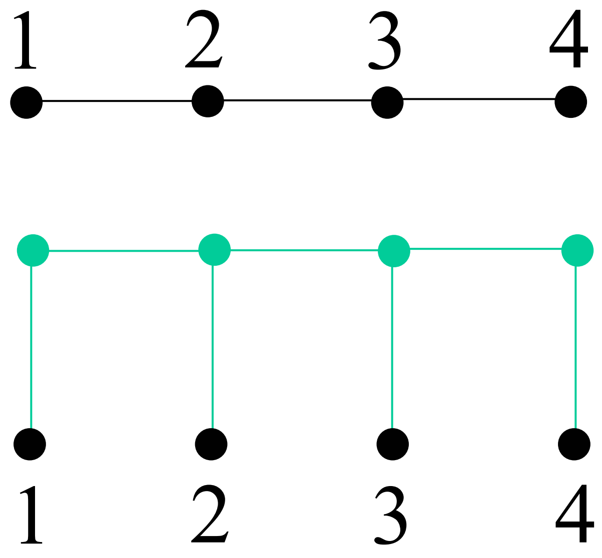
Leaf Powers

A graph is a *leaf power* if it is a k-leaf power for some $k \geq 2$.

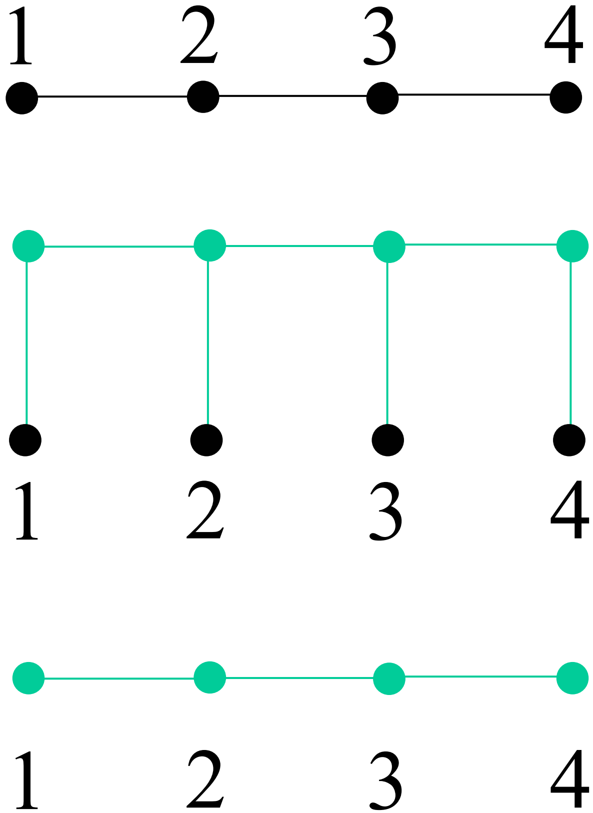
Obviously, the 2-leaf powers are exactly the disjoint unions of cliques.

Leaf Powers

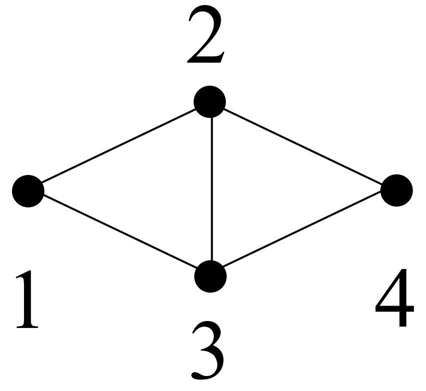
Leaf Powers



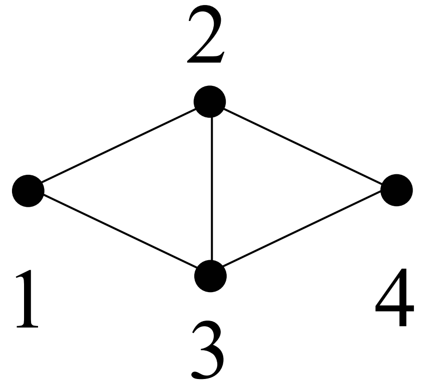
Leaf Powers

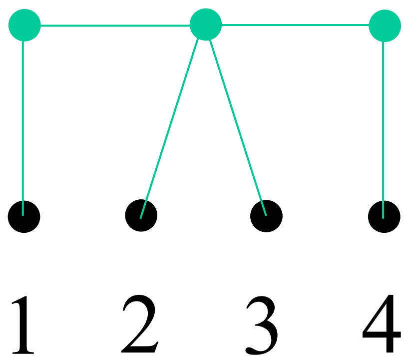


Leaf Powers



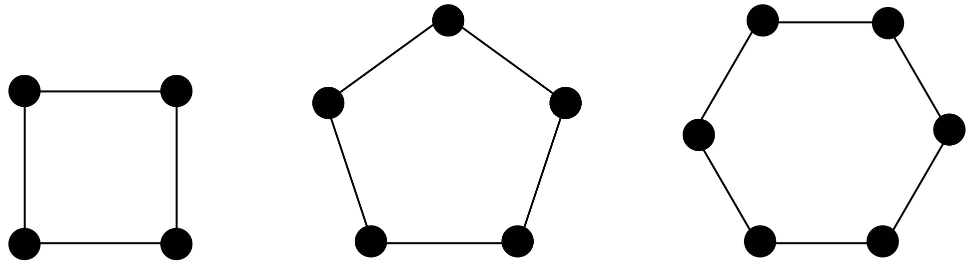
Leaf Powers





Chordal Graphs

Graph *G* is *chordal* if it contains no chordless cycles of length at least four.



Graph Powers

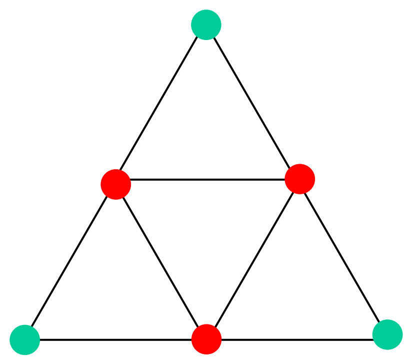
For graph $G = (V,E)$, let $G^k = (V, E^k)$ with

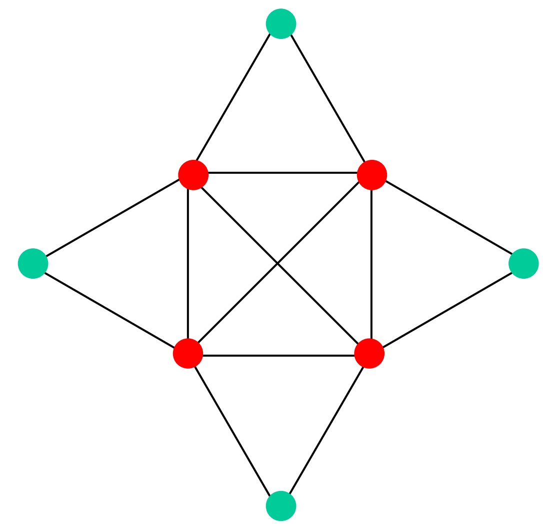
$$xy \in E^k \Leftrightarrow \text{dist}_G(x,y) \leq k$$

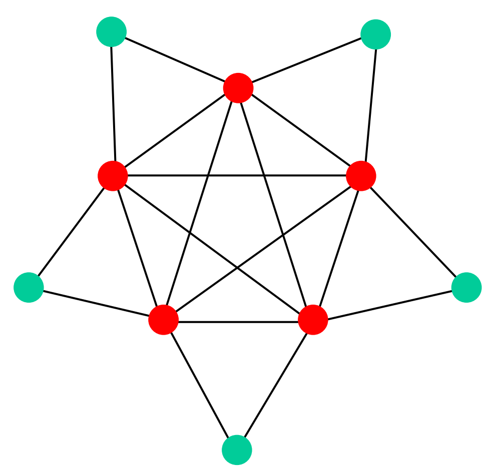
denote the *k-th power of G*.

Fact. Any k-leaf power is an induced subgraph of the k-th power of a tree. Any induced subgraph of a k-leaf power is a k-leaf power.

Fact. Powers of trees are chordal.







Leaf powers are strongly chordal

A graph is *strongly chordal* if it is chordal and sun-free. Trees are strongly chordal.

Theorem [Lubiw, 1982; Dahlhaus, Duchet, 1987; Raychaudhuri, 1992] For every $k \geq 2$:

G strongly chordal $\Rightarrow$ G^k strongly chordal.

Corollary For every $k \geq 2$, k-leaf powers are strongly chordal.

Neighborhood Subtree Tolerance Graphs

[Bibelnieks, Dearing, 1993, Neighborhood subtree tolerance graphs]:

NeST graphs are weakly chordal.

[Hayward, Kearney,1993; Hayward, Kearney, Malton, 2002]:

NeST graphs are weakly chordal but not vice versa.

Neighborhood Subtree Tolerance Graphs

Theorem [Hayward, Kearney,1993; Hayward, Kearney, Malton, 2002]:

A graph $G = (V,E)$ is a NeST graph with fixed tolerance $\Leftrightarrow$

$\exists$ positive constant $k > 0$ and an undirected weighted tree $T = (N,A,w)$ with $V \subseteq N$ and positive weights $w: A \rightarrow R$ such that

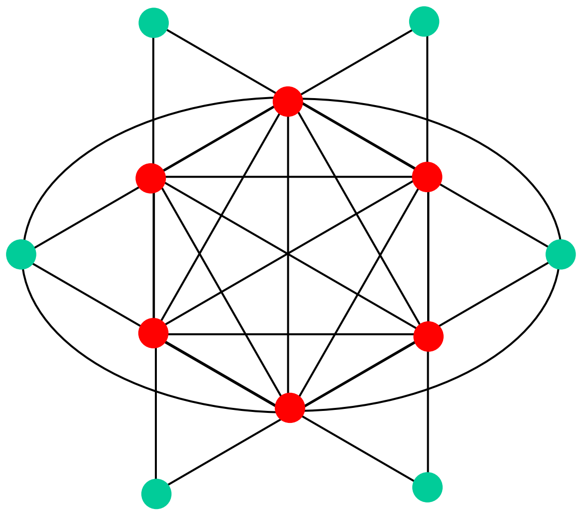
$$\forall\ u,v \in V\ (uv \in E \Leftrightarrow d_T\,(u,v) \leq k)$$

Neighborhood Subtree Tolerance Graphs

Fact. Fixed Tolerance NeST graphs are strongly chordal but not vice versa.

(proof is based on [Broin, Lowe, A dynamic programming algorithm for covering problems with (greedy) totally balanced constraint matrices, 1986])

No Fixed Tolerance NeST graph



Neighborhood Subtree Tolerance Graphs vs. Leaf Powers

Theorem [B., Hundt, Mancini, Wagner, 2008]:

G is a fixed tolerance NeST graph $\Leftrightarrow$

G is a leaf power.

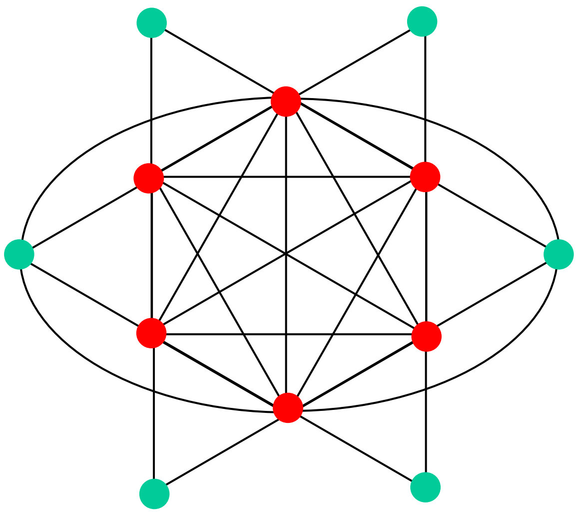
Corollary [B., Hundt, LATIN 2008]:

Interval graphs are leaf powers.

Theorem [B., Hundt, Mancini, Wagner, 2008]:

Rooted directed path graphs are leaf powers.

No Leaf Power



Inclusions

Let $L(k)$ denote the class of k-leaf powers.

Obviously, $L(k) \subset L(k+2)$ (by simply subdividing pendant edges in a k-leaf root).

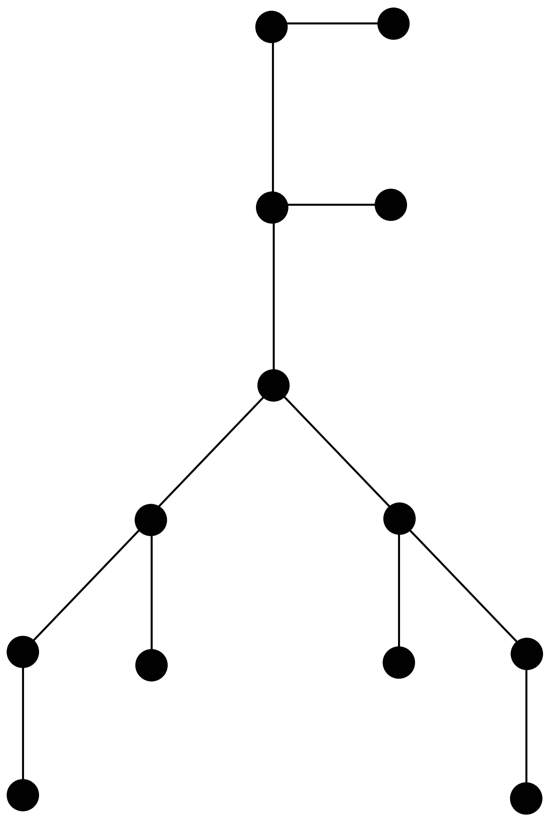
For a while, the following has been an open question:

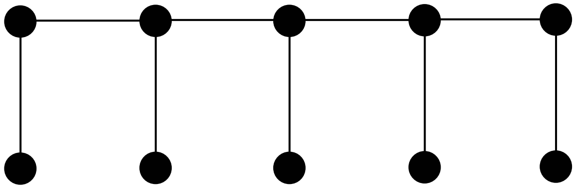
Is $L(k) \subset L(k+1)$?

Inclusions

[Fellows, Meister, Rosamond, Sritharan, Telle, 2007]:

$L(4) \not\subset L(5)$.





Inclusions

Theorem [Wagner, B. 2008] For any $2 \leq k < k'$

$L(k) \not\subset L(k') \Leftrightarrow k' \leq 2k - 3$ and $k' - k$ is odd.

Examples

$L(4) \subset L(6)$ and $L(4) \subset L(7)$ but $L(4) \not\subset L(5)$.

$L(5) \subset L(7)$ and $L(5) \subset L(8)$ but $L(5) \not\subset L(6)$.

$L(6) \subset L(8)$ and $L(6) \subset L(11)$ but

$L(6) \not\subset L(7)$ and $L(6) \not\subset L(9)$.

Inclusions

Corollary

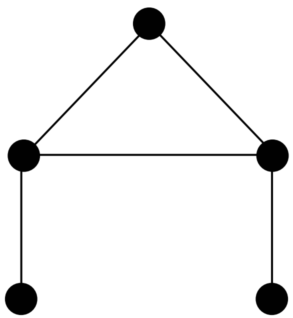
For all $k \geq 5$: $L(k) \not\subset L(k+1)$.

Corollary

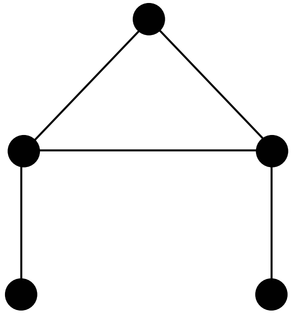
For $2 \leq k < k'$, the k-leaf power G is a k'-leaf power $\Leftrightarrow$

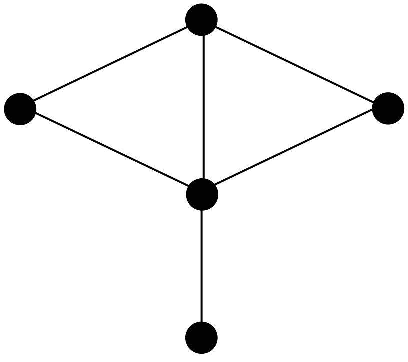
a k-leaf root T of G can be transformed to a k'-leaf root T' of G by subdividing edges of T.

3-Leaf Powers

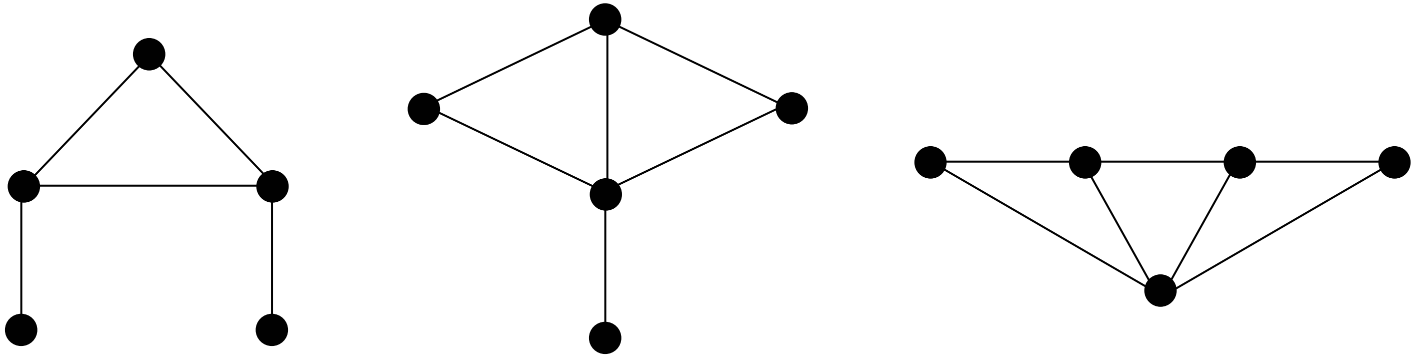


3-Leaf Powers

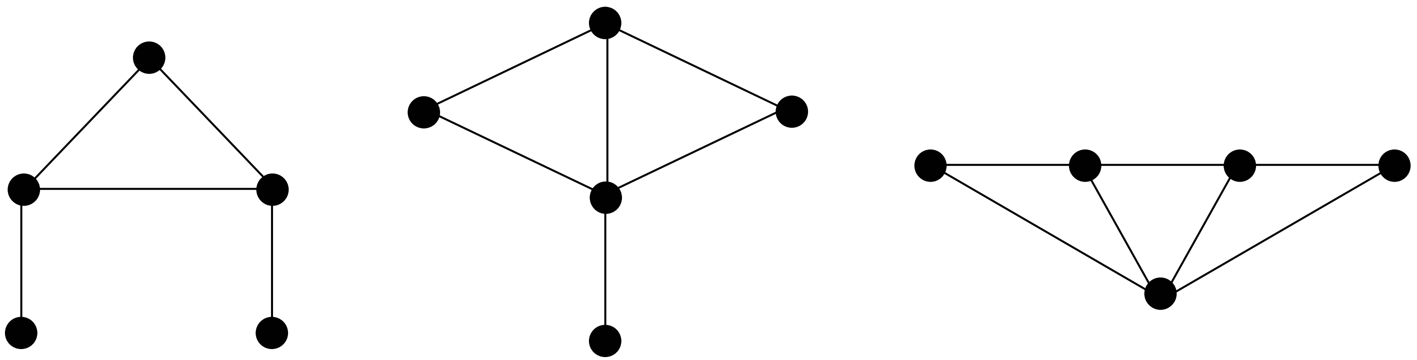




3-Leaf Powers



3-Leaf Powers



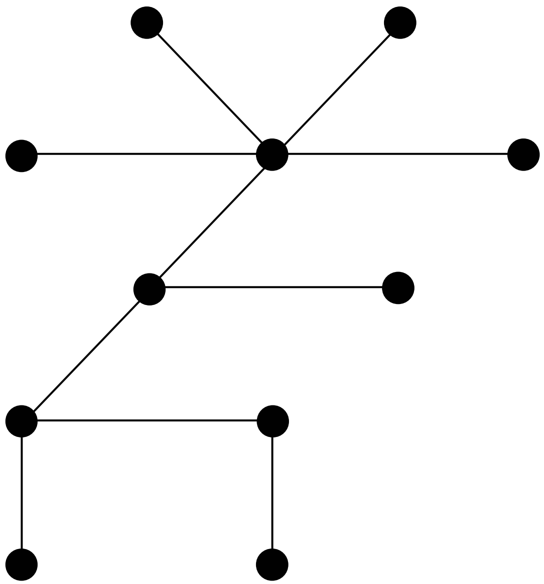
Theorem [Dom, Guo, Hüffner, Niedermeier, 2004]

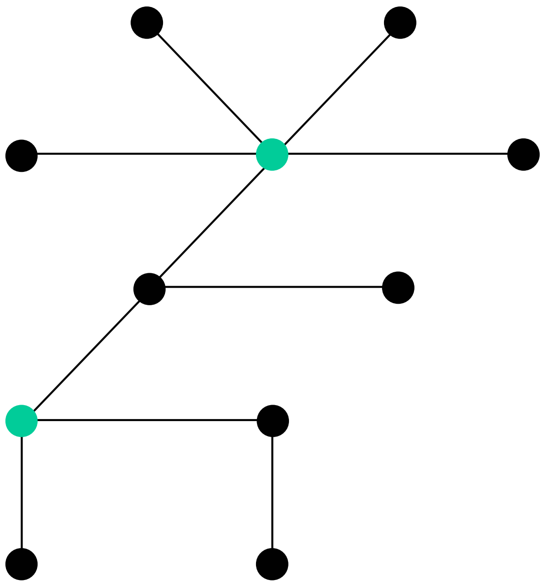
G is a 3-leaf power $\Leftrightarrow$ G is (bull, dart, gem)-free chordal $\Leftrightarrow$ critical clique graph of G is a tree.

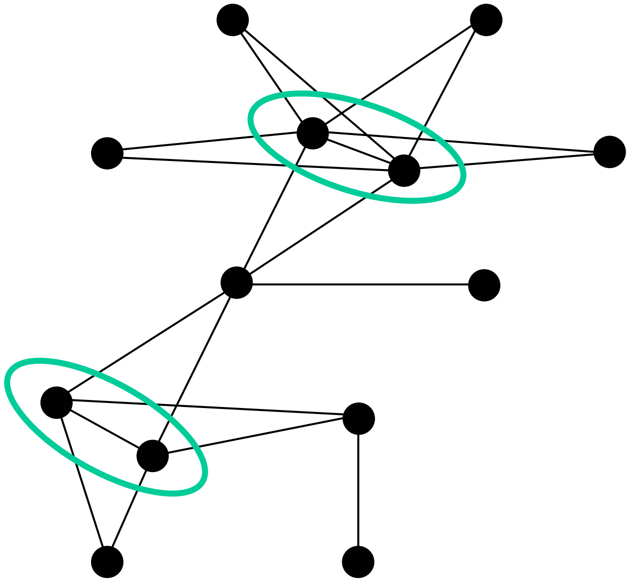
3-Leaf Powers

[B., Le, 2005; Rautenbach, 2004]

A connected graph G is a 3-leaf power $\Leftrightarrow$ G results from substituting cliques into the vertices of a tree.







3-Leaf Powers

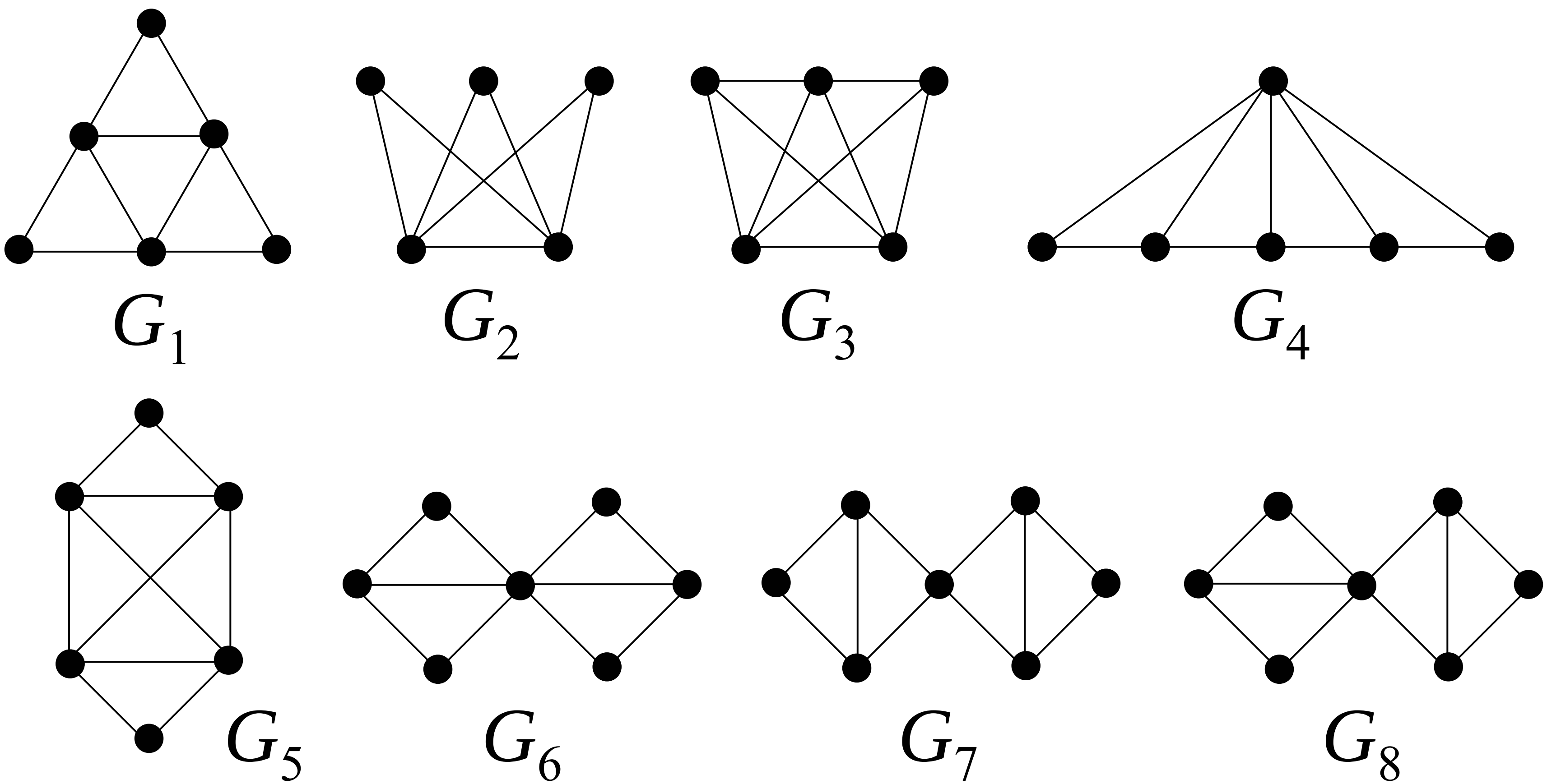
[B., Le, 2005; Rautenbach, 2004]

A connected graph G is a 3-leaf power $\Leftrightarrow$ G results from substituting cliques into the vertices of a tree.

[B., Le 2005]

Linear time recognition for 3-leaf powers.

4-Leaf Powers



4-Leaf Powers

Theorem [Rautenbach, 2004]

A graph G without true twins is a 4-leaf power $\Leftrightarrow$ G is $(G_1, \ldots, G_8)$-free chordal.

4-Leaf Powers

Theorem [B., Le, Sritharan, 2005]

For every graph *G*, the following conditions are equivalent:

(i) *G* is a 2-connected basic 4-leaf power.

(ii) *G* is the square of some tree.

(iii) *G* is chordal, 2-connected and (G_1, ..., G_5)-free.

4-Leaf Powers

Theorem [B., Le, Sritharan, 2005]

The following conditions are equivalent:

(i) *G* is a basic 4-leaf power.

(ii) Every block of *G* is the square of some tree, and for every non-disjoint pair of blocks, at least one of them is a clique.

(iii) *G* is an induced subgraph of the square of some tree.

(iv) *G* is $(G_1, \ldots, G_8)$-free chordal.

Distance-Hereditary 5-Leaf Powers

Theorem [B., Le, Rautenbach, 2006]

For every distance-hereditary graph G, the following conditions are equivalent:

(i) G is a basic 5-leaf power.

(ii) G is $(F_1, \ldots, F_{34})$–free chordal.

(iii) G fulfills “various“ block and gluing conditions ...

(k,l)-Leaf Powers

A finite undirected graph $G = (V,E)$ is a *(k,l)-leaf power* if there is a tree $T = (U, F)$ with leaf set V such that for all $x,y \in V$

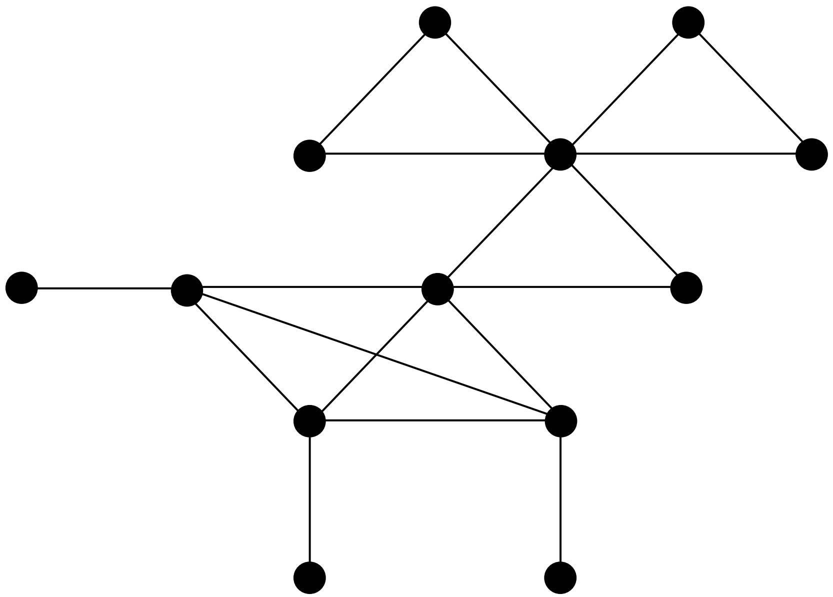
(i) $xy \in E \Rightarrow \mathrm{dist}_T(x,y) \leq k$ and

(ii) $xy \notin E \Rightarrow \mathrm{dist}_T(x,y) \geq l$.

Such a tree T is a *(k,l)-leaf root of G*.

Block Graphs

A (connected) graph *G* is a *block graph* if its blocks (i.e., 2-connected components) are cliques.



(4,6)-Leaf Powers

Theorem [B., Wagner, 2007]

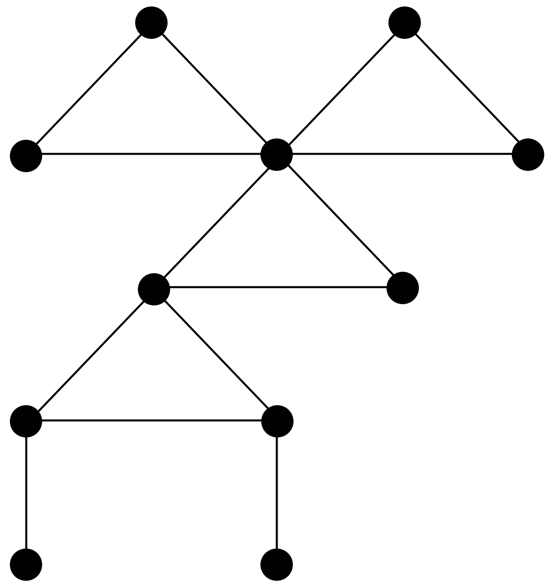
For connected graph G, the following are equivalent:

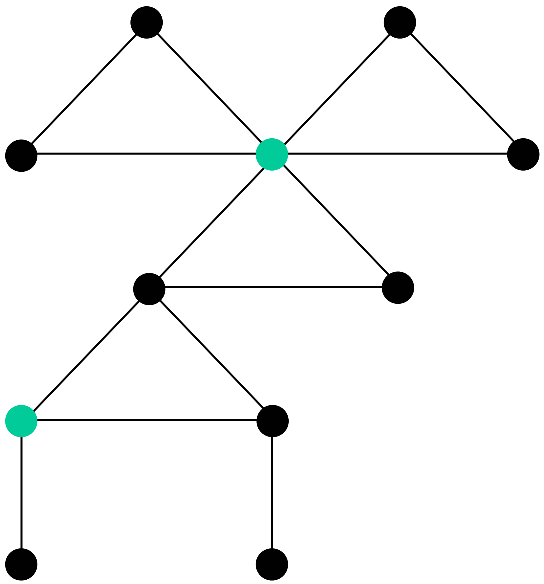
(i) G is a (4,6)-leaf power.

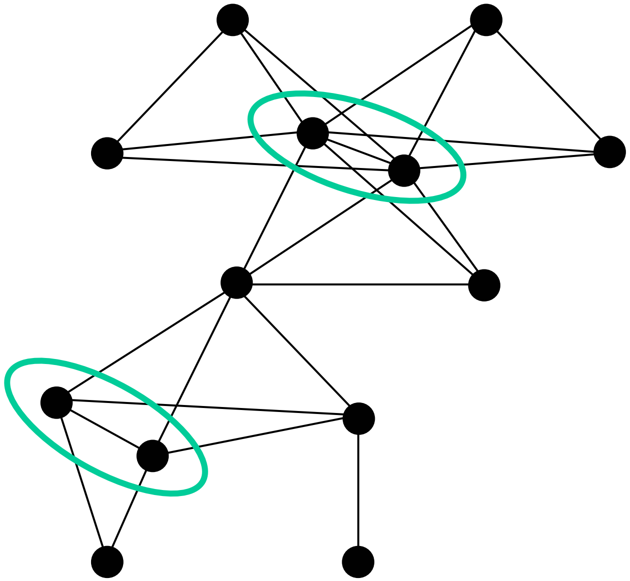
(ii) G is strictly chordal, i.e., (dart,gem)-free chordal.

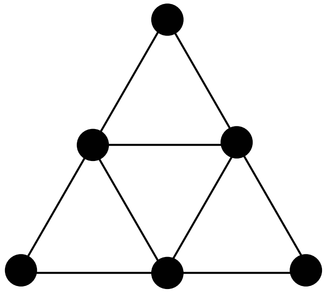
(iii) G results from a block graph by substituting cliques into its vertices.

(A paper by Kennedy, Lin and Yan 2006 shows that strictly chordal graphs are leaf powers.)

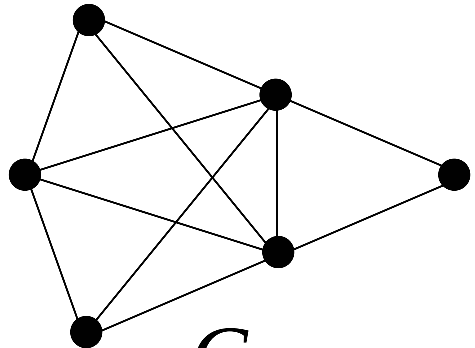




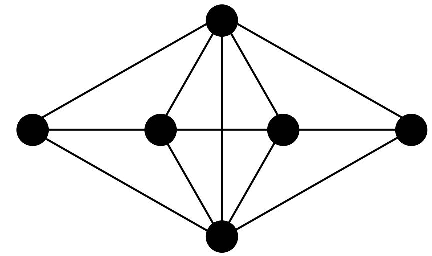




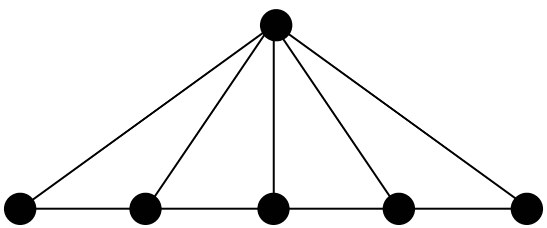
G_1



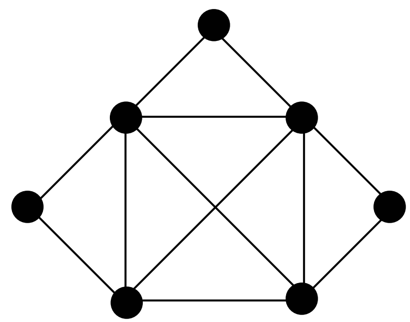
G_2



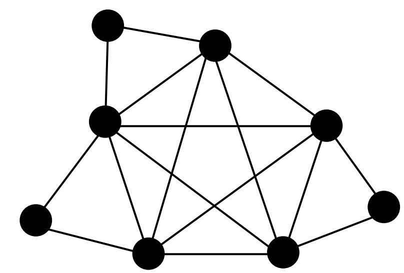
G_3



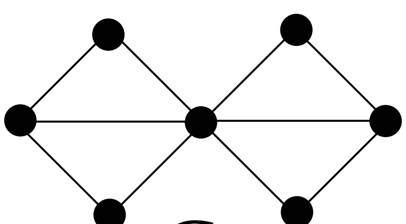
G_4



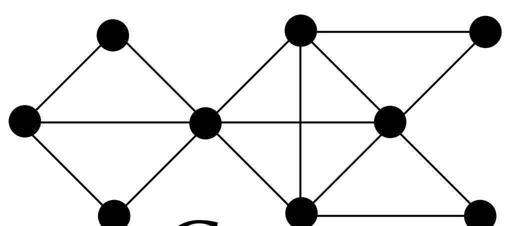
G_5



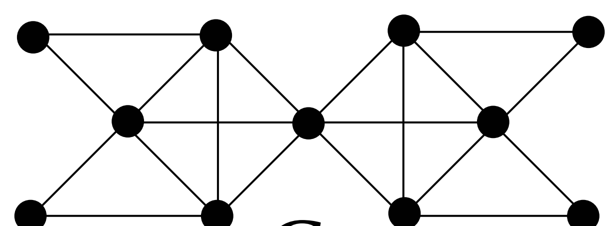
G_6



G_7



G_8



G_9

(6,8)-Leaf Powers

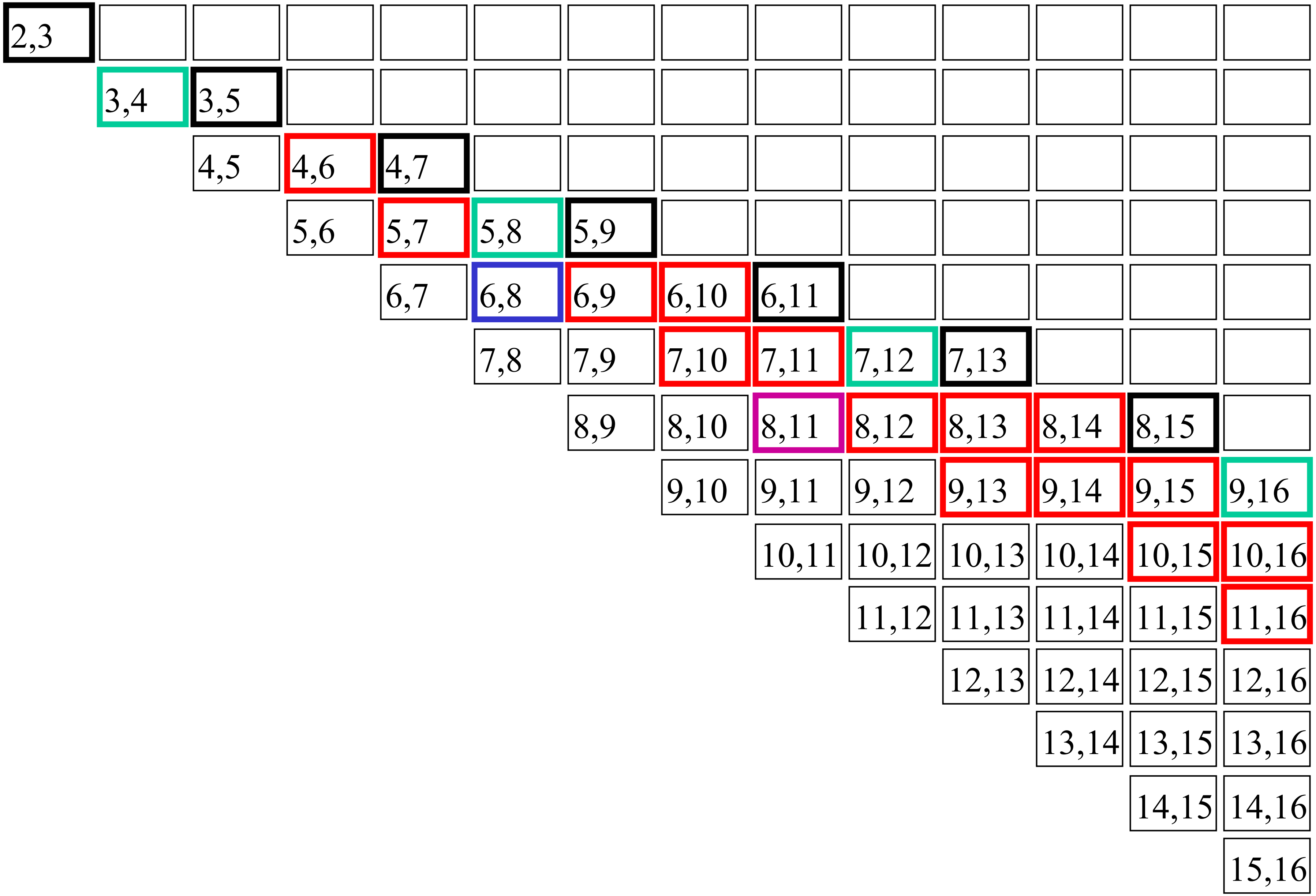
Theorem [B., Wagner, 2007]

The following conditions are equivalent:

(i) G is a basic (6,8)-leaf power.

(ii) G is an induced subgraph of the square of some block graph.

(iii) G is $(G_1, ..., G_9)$-free chordal.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8,9 8,10 8,11 8,12 8,13 8,14 8,15
9,10 9,11 9,12 9,13 9,14 9,15 9,16
10,11 10,12 10,13 10,14 10,15 10,16
11,12 11,13 11,14 11,15 11,16
12,13 12,14 12,15 12,16
13,14 13,15 13,16
14,15 14,16
15,16

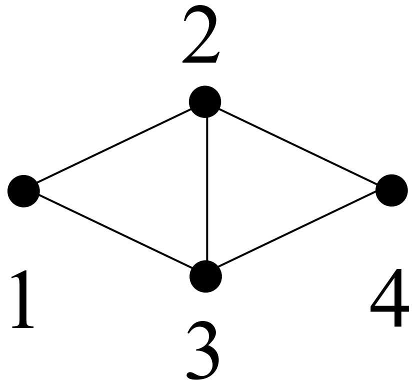
Simplicial Powers

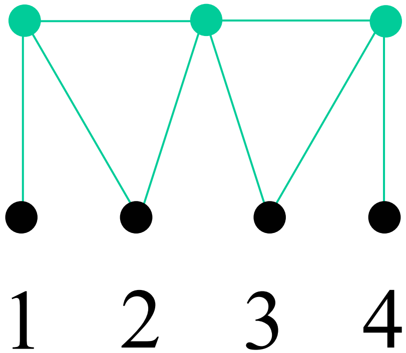
A finite undirected graph $G = (V_G , E_G)$ is the *k-simplicial power of a graph* $H = (V_H , E_H)$ if $V_G \subseteq V_H$ is the set of all simplicial vertices of H and for all distinct $x,y \in V_G$

$$xy \in E \Leftrightarrow \text{dist}_H (x,y) \leq k.$$

H is called a *k-simplicial root of G*.

Simplicial Powers





Trees vs. Block Graphs

The *line graph L(G)* of $G=(V,E)$ is the *edge intersection graph of G*, i.e., E is the set of nodes of $L(G)$, and distinct $e,e' \in E$ are adjacent in $L(G)$ if $e \cap e' \neq \emptyset$.

Theorem [Harary, 1972]

G is the line graph of a tree $\Leftrightarrow$

G is a claw-free block graph.

Trees vs. Block Graphs

Theorem [B., Le, 2008] For any $k \geq 2$,
G is the k-leaf power of a tree $\Leftrightarrow$
G is the $(k-1)$-simplicial power of a claw-free block graph.

Trees vs. Block Graphs

Buneman's 4-point condition:

For every four distinct vertices u, v, x, y,

the following holds:

(*) $\text{dist}_H(u,v) + \text{dist}_H(x,y) \leq \max\{$

$\text{dist}_H(u,x) + \text{dist}_H(v,y), \text{dist}_H(u,y) + \text{dist}_H(v,x)\}$

Trees vs. Block Graphs

Theorem [Buneman 1974, Howorka 1979]

Let G be a connected graph.

(i) G is a tree if and only if G is Δ-free and fulfills the 4-point condition (*).

(ii) G is a block graph if and only if G fulfills the 4-point condition (*).

Every graph is a simplicial power

Theorem

Every graph is the 2-simplicial power of a split graph.

Every graph is a simplicial power

Theorem

Every graph is the 2-simplicial power of a split graph.

2-SIMPLICIAL SPLIT GRAPH ROOT:

Instance: A graph G and an integer k.

Question: Is there a split graph H with $\leq k$ nodes such that G is the 2-simplicial power of H?

Every graph is a simplicial power

Theorem

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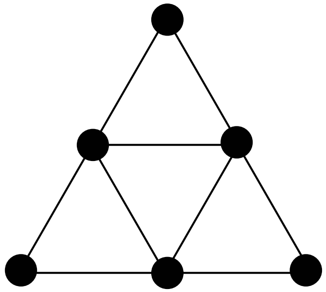
2-SIMPLICIAL SPLIT GRAPH ROOT:

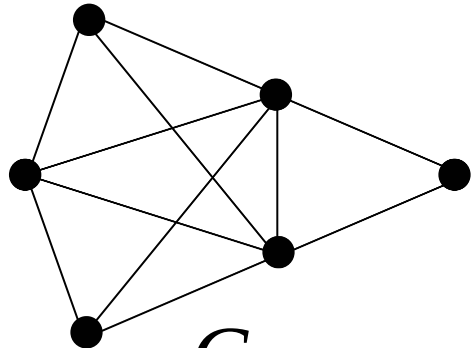
Instance: A graph G and an integer k.

Question: Is there a split graph H with $\leq k$ nodes such that G is the 2-simplicial power of H?

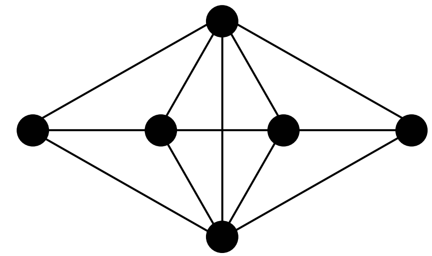
Theorem The problem 2-SIMPLICIAL SPLIT GRAPH ROOT is NP-complete.



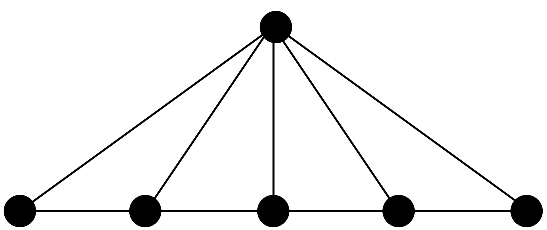
G_1



G_2



G_3



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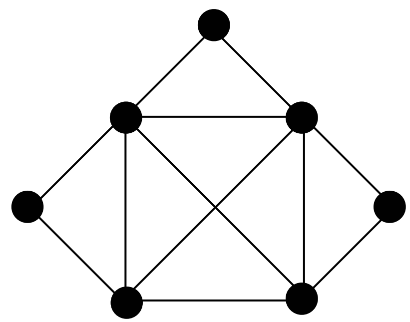

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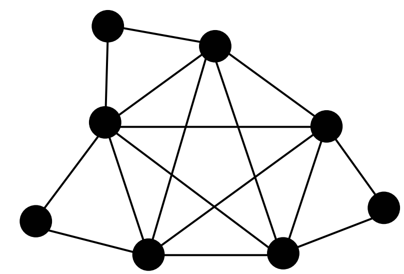

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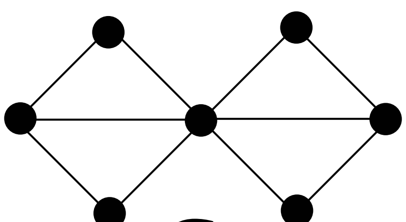
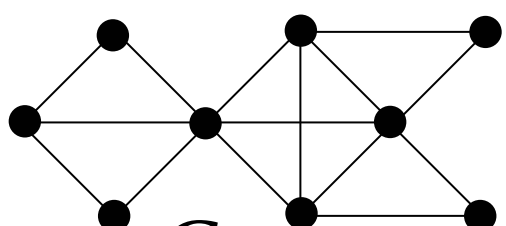
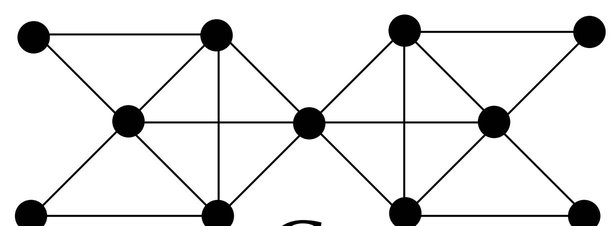


G_7



G_8



G_9

3-Simplicial Powers of Block Graphs

Theorem

The following conditions are equivalent:

(i) G is the 3-simplicial power of a block graph.

(ii) G is an induced subgraph of the square of a block graph.

(iii) Each block of G is a basic 3-simplicial power of a block graph, and each cut vertex v of G is non-special in at most one block containing v.

(iv) G is a basic (6,8)-leaf power.

(v) G is $(G_1, \ldots, G_9)$-free chordal.

Open Problems

Problem 1

Recognition complexity of leaf powers (of k-leaf powers for $k \geq 6$)

Problem 2

Complexity of determining the leaf rank of a given leaf power (even open for unit interval graphs)

Thank you for your attention!

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