Geometric Homomorphisms Part II

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Clinton, New York

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Outline

Definitions

Geochromatic Number, $X(\overline{G})$

$X(\overline{G}) \leq 4$

Duality

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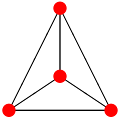

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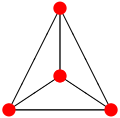

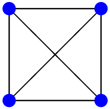

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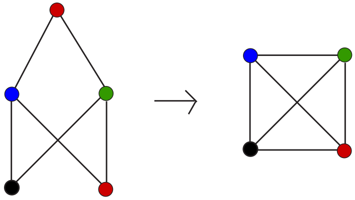

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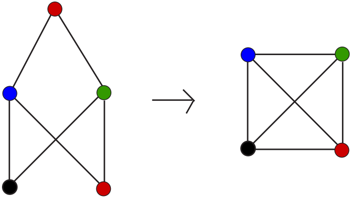

A homomorphism of geometric graphs will be assumed to be a geometric homomorphism.

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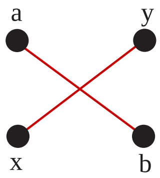

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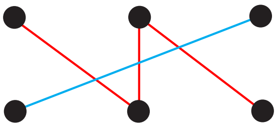

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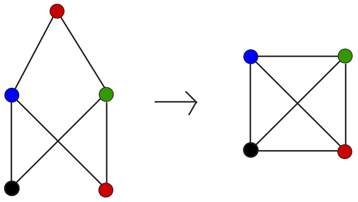

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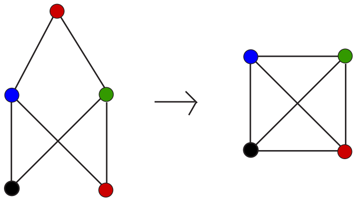

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However, there are 4-colorable, thickness-2 geometric graphs that are not geometrically homomorphic to .

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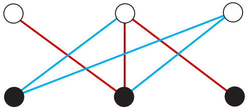

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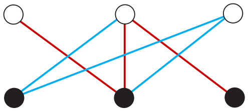

$$4 \neq X(\overline{G}) = 6$$

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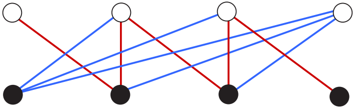

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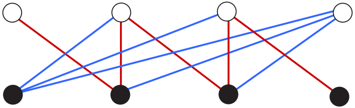

Since no pair of vertices can be identified, $X(\overline{G}) = n$.

Question

What else can we say about a geometric graph $\overline{G}$ where $X(\overline{G}) \leq 4$.

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First, some definitions.

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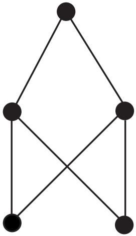
- the *crossing components of* $\overline{G}$ are connected components of the crossing subgraph;
- the *crossing component graph*, $C_{\times}$, is the graph with vertices associated with the crossing components of $\overline{G}$ and with edges between vertex pairs associated with components whose edges cross each other.



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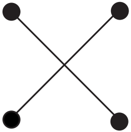
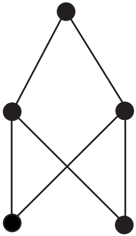


$\overline{G}_\times$, crossing subgraph



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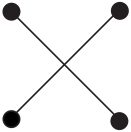

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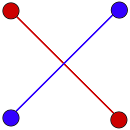

Crossing components

$C_{\times}$, crossing component graph

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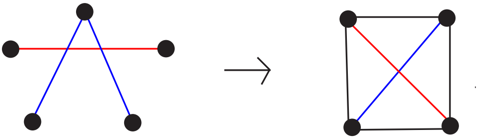

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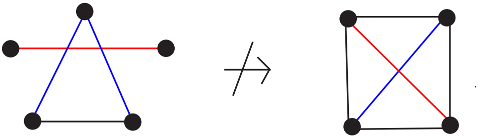

$\overline{G}[V_\times]$

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Notice the similarities to the conditions for $\overline{G} \rightarrow$  from Part I.

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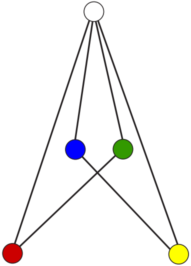

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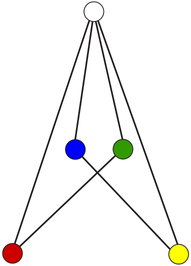

Fulfills Necessary Conditions, but no two vertices can be identified under a homomorphism $\Longrightarrow X(\overline{G}) = 5$
$\Longrightarrow \overline{G} \not\to \boxtimes$.

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We will apply the idea of duality to geometric homomorphisms.

Recall from Talk I

Thm: $\overline{G} \to \widehat{K}_{2,2} =$  $\iff$

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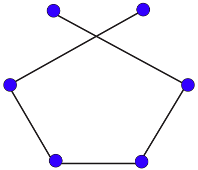

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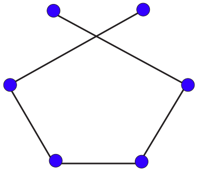

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Obs: $\widehat{P}_n \to \overline{G}_\times$ if and only if $\overline{G}$ has a crossing component that is not a planar embedding.

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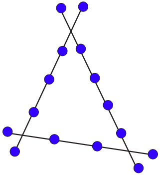

$\widehat{T}_{3,4,5}$

Obs: $\widehat{T}_{k_1,\ldots,k_{2r+1}} \to \overline{G}_\times \iff$ an odd cycle of crossing components in $\overline{G} \iff C_\times$ contains an odd cycle $\iff C_\times$ is not bipartite.

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By modifying $\widehat{P}_n$ and $\widehat{T}_{k_1,\ldots,k_{2s+1}}$ we could rewrite this theorem using $\overline{G}$ rather than $\overline{G}_\times$

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This does not capture the property that subgraphs induced by the crossing components in a color class of $C_\times$ must be bipartite. At the moment that does not seem feasible.

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Thank you!