Guarding a subgraph as a tool in pursuit-evasion games

Drago Bokal,
Gordana Radić

Department of mathematics and computer science
Faculty of natural sciences and mathematics
University of Maribor, Slovenia

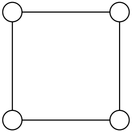
CANADAM, Montreal
May 2009

Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves

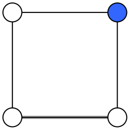
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



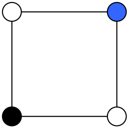
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



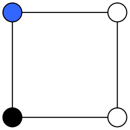
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



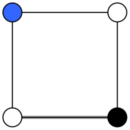
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



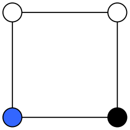
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



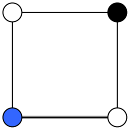
Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



Basic rules of the game

- played on a connected graph $G = (V, E)$
- 2 players: cop, robber
- perfect information
- cops move, robber moves



A single cop versus a single robber

- **cop-number**: $c(G)$

A single cop versus a single robber

- **cop-number**: $c(G)$

G is **cop-win** $\Leftrightarrow c(G) = 1$

A single cop versus a single robber

- **cop-number**: $c(G)$

G is **cop-win** $\Leftrightarrow c(G) = 1$

- K_n; $n \in \mathbb{N}$ is cop-win
- tree is cop-win
- C_n; $n \geq 4$ is robber-win

A single cop versus a single robber

- **cop-number**: $c(G)$

G is **cop-win** $\Leftrightarrow c(G) = 1$

- K_n; $n \in \mathbb{N}$ is cop-win
- tree is cop-win
- C_n; $n \geq 4$ is robber-win

R. Nowakowski, P. Winkler (1984):

G is cop-win $\Leftrightarrow$ G has **elimination order** of vertices.

Pursuit-evasion results I.

- M. Aigner, M. Fromme (1984):
  G **planar** $\Rightarrow c(G) \leq 3$

Pursuit-evasion results I.

- M. Aigner, M. Fromme (1984):
  G **planar** $\Rightarrow c(G) \leq 3$

- A. Quilliot (1985):
  genus of $G : \gamma(G) = k$
  $\Rightarrow c(G) \leq 3 + 2k$

Pursuit-evasion results I.

- M. Aigner, M. Fromme (1984):
  G **planar** $\Rightarrow c(G) \leq 3$

- A. Quilliot (1985):
  genus of $G : \gamma(G) = k$
  $\Rightarrow c(G) \leq 3 + 2k$

- P. Frankl (1987):
  girth of $G : g(G) \geq 8t - 3$
  $\Rightarrow c(G) > (\delta(G) - 1)^t$

Pursuit-evasion results II.

M. Aigner, M. Fromme (1984)

Let P be the **shortest path** between $u, v \in V(G)$. Then one cop guarantees that the robber will be immediatelly caught, if he moves onto P.

Some results:

S. Fitzpatrick, R. Nowakowski (2001):

$P = a_0 a_1 \ldots a_n$ an **shortest path** in G
$f : G \to P$ given by

$$f(v) = \begin{cases} a_k & ; \quad k = d(a_0, v) \text{ and } k \leq n \\ a_n & ; \quad \textit{otherwise} \end{cases}$$

Image or "shadow" of the robber will move along the path.

Some results:

S. Fitzpatrick, R. Nowakowski (2001):

$P = a_0 a_1 \dots a_n$ an **shortest path** in G
$f : G \to P$ given by

$$f(v) = \begin{cases} a_k & ; \quad k = d(a_0, v) \text{ and } k \leq n \\ a_n & ; \quad otherwise \end{cases}$$

Image or “shadow” of the robber will move along the path.

- $c(P) = 1 \Rightarrow$ Cop is able to **catch a shadow** and he can stay with it.

Some results:

S. Fitzpatrick, R. Nowakowski (2001):

$P = a_0 a_1 \ldots a_n$ an **shortest path** in G
$f : G \to P$ given by

$$f(v) = \begin{cases} a_k & ; \quad k = d(a_0, v) \text{ and } k \leq n \\ a_n & ; \quad \textit{otherwise} \end{cases}$$

Image or "shadow" of the robber will move along the path.

- $c(P) = 1 \Rightarrow$ Cop is able to **catch a shadow** and he can stay with it.
- Every vertex on the path is **its own shadow** $\Rightarrow$ robber will be **caught** on P.

Some results:

S. Fitzpatrick, R. Nowakowski (2001):

$P = a_0 a_1 \dots a_n$ an **shortest path** in G
$f : G \rightarrow P$ given by

$$f(v) = \begin{cases} a_k & ; \quad k = d(a_0, v) \text{ and } k \leq n \\ a_n & ; \quad \textit{otherwise} \end{cases}$$

Image or "shadow" of the robber will move along the path.

- $c(P) = 1 \Rightarrow$ Cop is able to **catch a shadow** and he can stay with it.
- Every vertex on the path is **its own shadow** $\Rightarrow$ robber will be **caught** on P.
- $H \subseteq G$, $f : G \rightarrow H$ a **retraction**, H cop-win. A single cop can catch the f-image of the robber in H and stay with it.

Types of guarding

- played on graph $G = (V, E)$
- for 2 players:
  - $i \ldots$ number of **intruders**
  - $g \ldots$ number of **guards**
- H subgraph for guarding

Types of guarding

- played on graph $G = (V, E)$
- for 2 players:

  $i \dots$ number of **intruders**

  $g \dots$ number of **guards**
- H subgraph for guarding

- first guards step onto G:
  guarding by **protection**.

Types of guarding

- played on graph $G = (V, E)$
- for 2 players:

  $i \ldots$ number of **intruders**

  $g \ldots$ number of **guards**
- H subgraph for guarding

- first guards step onto G: guarding by **protection**.
- first intruders step onto graph G: guarding by **pursuit-evasion**.

Types of guarding

- played on graph $G = (V, E)$
- for 2 players:
  $i \dots$ number of **intruders**
  $g \dots$ number of **guards**
- H subgraph for guarding

- first guards step onto G:
  guarding by **protection**.
- first intruders step onto graph G:
  guarding by **pursuit-evasion**.
- prevent guards stepping onto the intruder:
  guard-posts $S \subseteq V(G)$.

Guarding – formally

- played on graph $G = (V, E)$
- for 2 players:

  $i \ldots$ number of **intruders**

  $g \ldots$ number of **guards**
- H subgraph for guarding

Guarding – formally

- played on graph $G = (V, E)$
- for 2 players:
  $i \ldots$ number of **intruders**
  $g \ldots$ number of **guards**
- H subgraph for guarding

intruder: (v, ι)

- $v \in V(G)$
- $\iota : V^{i+g}(G) \to V(G)$
  $\iota(x, v_1, \ldots, v_{i+g-1}) \sim x$

Guarding – formally

- played on graph $G = (V, E)$
- for 2 players:

  $i \ldots$ number of **intruders**

  $g \ldots$ number of **guards**
- H subgraph for guarding

intruder: (v, ι)

- $v \in V(G)$
- $\iota : V^{i+g}(G) \to V(G)$
  $\iota(x, v_1, \ldots, v_{i+g-1}) \sim x$

guard: (φ, γ)

- $\varphi : V^i(G) \to S$
- $\gamma : V^{i+g}(G) \to V(G)$
  $\gamma(v_1, \ldots, v_{i+g-1}, y) \sim y$

The length of the game

Prosposition

$G \ldots$ graph $: n = |V(G)|$

$H \ldots$ subgraph $G : \widetilde{n} = |V(G) \backslash V(H)|$

$i \ldots$ number of intruders

$g \ldots$ number of guards

The intruders will enter $H \Leftrightarrow$ intruders will enter H in at most $2(n^g \widetilde{n}^i) - 1$ moves.

Notation

- G a **connected graph**
- H **subgraph** of G
- $S \subseteq V(G)$ a set of **guard posts**

Notation

- G a **connected graph**
- H **subgraph** of G
- $S \subseteq V(G)$ a set of **guard posts**

$W = x_0 \dots x_l$ a **walk** in G
$Y = (\varphi, \gamma)$ a **guard** in G

Position y_i of the guard after following the intruder on W:

$y_0(W, Y) := \varphi(x_0)$
$y_i(W, Y) := \gamma(x_i, y_{i-1})$

Notation

- G a **connected graph**
- H **subgraph** of G
- $S \subseteq V(G)$ a set of **guard posts**

$W = x_0 \dots x_l$ a **walk** in G
$Y = (\varphi, \gamma)$ a **guard** in G

Position y_i of the guard after following the intruder on W:

$$y_0(W, Y) := \varphi(x_0)$$
$$y_i(W, Y) := \gamma(x_i, y_{i-1})$$

All forced positions of Y:

$$\Phi_Y(x) := \{y_l(W, Y) \mid W = x_1 \dots x_{l-1}x\}$$

W runs over **all finite walks** in $G - H$ which end in x.

Defending against vertices

Assume: $W = x \dots x' \quad x' \in V(H)$

Defending against vertices

Assume: $W = x \dots x' \quad x' \in V(H)$

- A vertex $y \in V(G)$ is said to **parry** $x \in V(G)$ **against the walk** W, if there exists some walk W' of length $|W|$ from y to x'. Notation: $y \succeq_W x$

Defending against vertices

Assume: $W = x \dots x' \quad x' \in V(H)$

- A vertex $y \in V(G)$ is said to **parry** $x \in V(G)$ **against the walk** W, if there exists some walk W' of length $|W|$ from y to x'. Notation: $y \succeq_W x$
- If $y \in V(G)$ parries $x \in V(G)$ against any walk W from x to any $x' \in V(H)$, then y **parries** x. Notation: $y \succeq x$.

Defending against vertices

Assume: $W = x \dots x' \quad x' \in V(H)$

- A vertex $y \in V(G)$ is said to **parry** $x \in V(G)$ **against the walk** W, if there exists some walk W' of length $|W|$ from y to x'. Notation: $y \succeq_W x$
- If $y \in V(G)$ parries $x \in V(G)$ against any walk W from x to any $x' \in V(H)$, then y **parries** x. Notation: $y \succeq x$.
- Let $Q : V(G) \rightarrow 2^{V(G)}$ be a function

$$Q(x) := \{y \in V(G) \,|\, \forall z \in H : d(y,z) \leq d(x,z)\}.$$
$$Q : V(G) \rightarrow 2^{V(G)} \text{ can be computed in } O(n^2 m).$$

Some properties of $\succeq$

- Reflexive: $\forall x$: $x \succeq x$.

Some properties of $\succeq$

- Reflexive: $\forall x$: $x \succeq x$.
- Transitive: $\forall x, y, z$: $z \succeq y$ and $y \succeq x$ implies $z \succeq x$.

Some properties of $\succeq$

- Reflexive: $\forall x$: $x \succeq x$.
- Transitive: $\forall x, y, z$: $z \succeq y$ and $y \succeq x$ implies $z \succeq x$.
- Not antisymmetric: $y \succeq x$ and $x \succeq y$ is possible for $x \neq y$.

Some properties of $\succeq$

- Reflexive: $\forall x$: $x \succeq x$.
- Transitive: $\forall x, y, z$: $z \succeq y$ and $y \succeq x$ implies $z \succeq x$.
- Not antisymmetric: $y \succeq x$ and $x \succeq y$ is possible for $x \neq y$.
- $\succeq$ is a preorder.

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^{+}(x')$ for any $xx' \in E(G)$.

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^{+}(x')$ for any $xx' \in E(G)$.

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^+(x')$ for any $xx' \in E(G)$.

Proof.

$\Rightarrow$: Y a successful guard: Φ_Y satisfies these properties. $\Leftarrow$: $y \in \Phi(x) \Rightarrow y \succeq x$. Intruder on x. Initially, the cop moves into $\Phi(x) \cap S$, and then stays with $\Phi(x)$. □

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^{+}(x')$ for any $xx' \in E(G)$.

Proof.

$\Rightarrow$: Y a successful guard: Φ_Y satisfies these properties. $\Leftarrow$: $y \in \Phi(x) \Rightarrow y \succeq x$. Intruder on x. Initially, the cop moves into $\Phi(x) \cap S$, and then stays with $\Phi(x)$. □

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^{+}(x')$ for any $xx' \in E(G)$.

Proof.

$\Rightarrow$: Y a successful guard: Φ_Y satisfies these properties. $\Leftarrow$: $y \in \Phi(x) \Rightarrow y \succeq x$. Intruder on x. Initially, the cop moves into $\Phi(x) \cap S$, and then stays with $\Phi(x)$. □

A classification of guarded subgraphs

Theorem

Let G be a graph, H its subgraph and S the set of guard posts. Then H can be guarded in G with **a single guard against a single intruder**, if and only if there exists a function $\Phi : V(G) \to 2^{V(G)}$, such that

- $\Phi(x) \subseteq Q(x)$ for any $x \in V(G)$,
- $\Phi(x) \cap S \neq \varnothing$ for any $x \in V(G)$, and
- $\Phi(x) \subseteq \Phi^{+}(x')$ for any $xx' \in E(G)$.

Proof.

$\Rightarrow$: Y a successful guard: Φ_Y satisfies these properties. $\Leftarrow$: $y \in \Phi(x) \Rightarrow y \succeq x$. Intruder on x. Initially, the cop moves into $\Phi(x) \cap S$, and then stays with $\Phi(x)$. □

The shadow function

Definition

Inclusion maximal function Φ is called the **shadow function** *of H in G.*

The shadow function

Definition

Inclusion maximal function Φ is called the **shadow function** *of H in G.*

Theorem

Let $\Phi(x) = \{y \mid y \succeq x\}$. Then Φ is the inclusion-maximal function satisfying the properties from the theorem and can be computed in $O(nm^2)$.

The intruding function

Definition

Let Φ be the shadow function of H in G. Define $\Psi : V(G) \times V(G) \rightarrow 2^{V(G)}$ *as* $\Psi(x, y) = \{z \in N_G(x) \mid \Phi(z) \cap N_G(y) = \emptyset\}$. *$\Psi$ is called the* **intruding function** *of H in G.*

The intruding function

Definition

Let Φ be the shadow function of H in G. Define $\Psi : V(G) \times V(G) \to 2^{V(G)}$ as $\Psi(x, y) = \{z \in N_G(x) \mid \Phi(z) \cap N_G(y) = \emptyset\}$. Ψ is called the **intruding function** *of H in G.*

Theorem

The following are equivalent for $x, y \in V(G)$:

- $y \succeq x$,
- $y \in \Phi(x)$,
- $\Psi(x, y) = \emptyset$.

Correspondences

- G, H, x – define the shadow $\Phi(x)$ of x.

Correspondences

- G, H, x – define the shadow $\Phi(x)$ of x.
- G, H, x, y – define the intruding function $\Psi(x, y)$.

Correspondences

- G, H, x – define the shadow $\Phi(x)$ of x.
- G, H, x, y – define the intruding function $\Psi(x, y)$.
- G, H, x, y – define where the cop should go from y, $\Phi(x) \cap N(y)$.

Correspondences

- G, H, x – define the shadow $\Phi(x)$ of x.
- G, H, x, y – define the intruding function $\Psi(x, y)$.
- G, H, x, y – define where the cop should go from y, $\Phi(x) \cap N(y)$.
- **Retraction** – cop's move depends only on the position of the robber?

The shadow and retractions

Reminder

$H \subseteq G$, $f : G \to H$ a homomorphism with $f/H = \mathrm{id}$. Then f is a retraction.

The shadow and retractions

Reminder

$H \subseteq G$, $f : G \to H$ a homomorphism with $f / H = \text{id}$. Then f is a retraction.

Lema

$H \subseteq G$, $f : G \to H$ a retraction. Then $f(x) \in \Phi(x)$ for all $x \in V(G)$.

The shadow and retractions

Reminder

$H \subseteq G$, $f : G \to H$ a homomorphism with $f/H = \mathrm{id}$. Then f is a retraction.

Lema

$H \subseteq G$, $f : G \to H$ a retraction. Then $f(x) \in \Phi(x)$ for all $x \in V(G)$.

Problem

$H \subseteq G$, $S \subseteq V(G)$ a set of guard posts. H can be guarded using S, if and only if there is a H', $H \subseteq H' \subseteq G[S]$, and a retraction of G onto H'.

Thank you for your attention!